**Mathematics 503 Complex Analysis Fall 2017**

## Week 2 workshop problems

For problems for consideration during the workshop I prefer to have problems to discuss besides the weekly assigned homework problems. If time is left over after discussing these workshop problems the homework problems due that week can be discussed.

1. A slightly more symmetric version of summation by parts is

$$\sum_l^m fn(g_{n+1} - g_n) + \sum_l^m g_{n+1}(f_{n+1} - f_n) = f_{m+1}g_{m+1} - f_l g_l$$

reflecting that the usual integration of parts is related to integrating the chain rule $(fg' + f'g = (fg)')$. Take $b_n = g_{n+1} - g_n, a_n = f_n$ to get the text's version in exercise I.14.

a) Use partial summation to show the first equality in

$$\sum_1^\infty \frac{1 + 1/2 + 13 + \cdots 1/n}{1 + 2 + 3 + \cdots n} = \sum_1^\infty \frac{2}{n^2} = \pi^2/3.$$

We will prove the second equality later in the course.

b) Prove Dirichlet's generalization of the alternating series test: Let $y_n \in \mathbf{C}$ be such that $\sum_1^k y_n$ are bounded for all $k$, and $x_n \in \mathbf{R}$ are monotone and have limit 0. Show $\sum_1^\infty x_k yk$ converges by applying partial summation. Show that if $|z| <= 1$ and $z \neq 1$ then $\sum_1^\infty z^n/n$ converges.

c) Let $y_n \in \mathbf{C}$ be such that $\sum_1^\infty y_n$ converges for all $k$, and $x_n \in \mathbf{R}$ are monotone and bounded. Show $\sum_1^\infty x_k yk$ converges.

2. Generalize Abel's Limit Theorem (exercise I.15) to show the following: suppose $\sum_1^\infty a_n$ converges. Then $\lim_{z\to 1} \sum_0^\infty a_n z^n = \sum a_n$ when $z$ approaches 1 such that $|1-z|/(1-|z|)$ is bounded. Hint: By altering $a_0$ by subtracting $\sum_0^\infty a_n$ we can assume $\sum_0^\infty a_n = 0$. Use partial summation to show $\sum_0^\infty a_n z^n = (1-z)\sum_0^\infty s_n z^n$ where $s_n$ are the partial sums of $\sum a_n$. Bound the tail of $\sum_0^\infty s_n |z^n|$ by a geometric series using $z$ is in the given region and take the limit as $z \to 1$ in the expresssion $(1-z)\sum_0^\infty s_n z^n$.

3. Apply Cauchy's theorem to the function

$$e^{e^{iz}}/z$$

and the portion of an annulus between circles of radius $R$ and $\epsilon$ in the first quadrant to determine (as Cauchy did) the value of

$$\int_0^\infty \frac{e^{cos(x)} sin(sin(x))}{x}\, dx$$

4. Apply Cauchy's theorem to rectangles with vertices $(-T,0),(T,0),(T,1),(-T,1)$ with circles of radius $\epsilon$ centered at $0, 2\pi i$ removed to show that for $0 < a < 1$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{(1-e^{x})}\,dx = \frac{\pi}{\tan(a\pi)}$$

Be sure to explain what you mean by the improper integral above.