Mathematics 503 **Complex Analysis** **Fall 2017**

Week 10 workshop problems

1. (Spring 2008 Rutgers Qualifying exam) Does there exist a holomorphic map of the right half plane to itself which takes 1 to 1 and 2 to 4 ?

2. Let $\mathcal{F}_a$ be the set of functions $f(z)$ holomorphic in the strip $\Im(z) < a$ such that $|f(x+iy)| \leq A/(1+x^2)$ in the strip for some constant A. Show that if $f(z) \in \mathcal{F}_a$ then $f'(z) \in \mathcal{F}_b$ for all $b < a$ by using an appropriate Cauchy estimate for the derivative, and the fact that for fixed $r > 0$, and all real x, $(1 + (x - r)^2) > D(1 + x^2)$ for some constant D.

3. Let $f(z)$ be holomorphic on the strip $\sigma_1 < \Re(z) < \sigma_2$ and continuous on the closure of the strip . Assume that as $|z| \to \infty$ with z in the strip that $f(z) \to 0$, and that $|f(z)| \leq 1$ on the sides of the strip. Show that $f(z)| \leq 1$ on the strip.

4. Let $f(z)$ be holomorphic in the right open half plane $\Re(z) > 0$, and continuous on $\Re(z) \geq 0$. Suppose that $|f(z)| \leq 1$ when $\Re(z) = 0$.

   a) Show by example that $|f(z)|$ can be unbounded on the right half plane.

   b) Show that if there exist a real number $\alpha < 1$ and constants A, B such that $|f(z)| \leq Ae^{B|z|^\alpha}$ when $\Re(z) > 0$ then $|f(z)| \leq 1$ on $Re(z) > 0$. Hint: use the Phragmen-Lindelof idea of multiplying $f(z)$ by a damping function $e^{-\epsilon z^\beta}$ with $\alpha < \beta < 1$ and using the maximal principle on half disks in the half plane, and then letting $\epsilon \to 0$.