Mathematics 503 Complex Analysis Fall 2017

Using the residue formula to compute real integrals

The birth of the residue formula was tied to Cauchy's desire to compute integrals of real functions. There are several classes of real functions that can be integrated over particular intervals by an application of the residue formula, using a rectangular, circular or semicircular contour. The sections below describe hypotheses that guarantee that integrals over certain arcs of contours will approach 0, so that the residue theorem applies to give information about real integrals. The main observation is that when a function satisfying $\lim_{|z|\to\infty} zf(z) = 0$ is integrated on a path γ_R of distance d_R from the origin having length at most a constant times R, the integral will approach 0 as d_R increases without bound.

For each type of integral examples are given. Such integrals are perennial exam questions. A notation such as F14 refers to the Rutgers Mathematics Department qualifying exam in Fall 14.

1. Suppose that $F(z)$ is a complex function on an open set about the unit disk which is holomorphic except for a finite number of points in the interior of the disk. Using the definition of the trigonometric functions give

$$\int_0^{2\pi} F(\cos t, \sin t)\, dt = -i \int_{|z|=1} F((z+1/z)/2, (z-1/z)/(2i))\, dz/z$$

When the residue theorem applies to the integrand we obtain that

$$\int_0^{2\pi} F(\cos t, \sin t)\, dt = 2\pi \sum_{z_i} Res_{z_i} 1/z)h(z)$$

Examples:

$$\int_0^{2\pi} dt/(a+\cos t) = 2\pi/\sqrt{(}a^2-1) \text{ if} a > 1$$

$$\int_0^{2\pi} dt/(a+\cos t)^2 = 2\pi a/\sqrt{(a^2-1)^3)} \text{ if} a > 1$$

$$\int_0^{2\pi/} dt/(5-3\sin t)^2 = 5\pi/32$$

QF16,QF15,QF12

2. Let $f(z)$ be holomorphic in an open set containing the closed upper half plane, except for a finite set of points z_i none of which are real. If $|zf(z)|$ approaches zero as $|z| \to \infty$ then $f(z)$ is integrable on $\mathbf{R}$ and

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum Res_{z_i}(f(z))$$

the sum taken over residues in the open upper half plane.

Examples:

$$\int_{-\infty}^{\infty} x^2/(1+x^4) = \pi/sqrt(2)$$

$$\int_{-\infty}^{\infty} x^{m-1}/(1+x^n) = 2\pi/(n\sin(m/n\pi))$$

QS13,QF12

3. Fourier transforms $\int_{-\infty}^{\infty} g(x)e^{iyx}$. When $g(z)$ is holomorphic on an open set containing the upper half plane except for a finite number of points $z_1, \ldots, z_k$ which are not real numbers, and $\lim_{z\to\infty} g(z) = 0$ we have

$$\int_{-\infty}^{\infty} g(x)e^{iax} = 2\pi i sign(a) \sum Res_{z_i}(g(z)e^{iaz})$$

Examples:

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x-ib}\, dx = 2\pi i e^{-ab}, a > 0, Re(b) > 0$$

QS16,QF14,QS14

4. Mellin transforms $\int_0^{\infty} q(x)x^a dx/x$. Assume $q(x)$ is holomorphic in the complement of $[0,\infty)$ except at points $z_1, \ldots, z_k$. Suppose further that $lim_{z\to 0} q(z)z^a = 0 = lim_{z\to\infty} q(z)z^a$ where $z^a = e^(alogz)$ where log is the branch given by $logre^{i\theta} = logr + i\theta$ where $\theta \in [0, 2\pi)$. When a is not an integer

$$\int_0^{\infty} q(x)x^a dx/x = \frac{2\pi i}{1-e^{2\pi i a}} \sum Res_{z_i}(q(z)z^{a-1})$$

Examples:

$$\int_0^{\infty} \frac{x^{-1}}{x+e^{i\phi}} = \frac{\pi}{sin\pi a} e^{i(a-1)\phi}, 0 < a < 1, -\pi < \phi < \pi$$

QF13

6. Sums of $f(n)$ or $(-1)^n f(n)$ can be reduced down to residue computations in some situations.

   a) Let $f(z)$ satisfy that $lim_{z\to\infty} zf(z) = 0$ and that $f(z)$ is holomorphic on $\mathbf{C}$ except at $z_1, \ldots, z_k$. Then

$$\sum_{n\in\mathbf{Z}, n\neq z_i} f(n) = -\sum Res_{z_i}(f(z)\cot(\pi z))$$

b) Let $f(z)$ satisfy that $lim_{z\to\infty} f(z) = 0$ and that $f(z)$ is holomorphic on $\mathbf{C}$ except at $z_1, \ldots, z_k$. Then

$$\sum_{n\in\mathbf{Z}, n\neq z_i} (-1)^n f(n) = -\sum Res_{z_i}(f(z)\csc(\pi z))$$

These can be proved using rectangles with vertices at $(N+1/2, iN), (-N-1/2, iN)$ and their conjugates and noting that $\cot(\pi z)$ is bounded on the sides of such rectangles (independent of N) and that $\csc(\pi z)$ bounded by a constant times $e^{-|y|}$ on $1/2+iy$ and by a constant times e^{-N} on $x+iN$.

c) Examples:

$$\sum 1/n^{2k} = \frac{-(2\pi)^{2k} B_{2k}(-1)^k}{2(2k)!}$$

where $z/(e^z - 1) = \sum B_k/k! z^k$.