Mathematics 503 Complex Analysis Fall 2017

Brief notes on homotopy and path integrals

Let $\alpha(t), \beta(t), t \in [0,1]$ be two continuous paths from z_0 to z_1 which lie in an connected open set Ω. We say that they are homotopic if there exists a continuous function $H(t,u) : [0,1] \times [0,1] \to \Omega$ such that $H(t,0) = \alpha(t), H(t,1) = \beta(t), H(0,u) = z_0, H(1,u) = z_1$. Intuitively, the homotopy deforms the path α to the path β, leaving the endpoints z_0, z_1 fixed. We write $\alpha \sim \beta$ to indicate homotopic paths.

Theorem: Let $f(z)$ be a holomorphic function on an open subset Ω of the complex plane. Suppose that α, β are homotopic paths in Ω. Then

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$

To prove this we will prove a slightly more general result:

Proposition: Let $f(z)$ be a holomorphic function on an open subset Ω of the complex plane. Let $H : [0,1] \times [0,1] \to \Omega$ be continuous. Let α be the path $H(t,0)$ followed by $H(1,u)$. Similarly let β be the path $H(0,u)$ followed by $H(t,1)$. Then $\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz$.

If we denote by γ the image of the boundary of the square, the proposition is equivalent to $\int_\gamma f(z)\, dz = 0$.

The proposition clearly implies the theorem. The idea of the proof of the proposition is to divide the unit square into smaller squares having the property that the image of the small squares are contained in open disks contained in Ω. By Cauchy's Theorem in the disk each integral over the boundary of the small squares is 0. Adding these together and noting the cancellation of integrals on segments shared by two squares will establish the result of the proposition.

We need two properties of continuous maps on a compact subset of a metric space to another metric space.

A) If K is a compact subset contained in an open subset U of a metric space Y then there exists an $\epsilon > 0$ such that for all $x \in K$ the ϵ-ball $\{y \in Y | d(x,y) < \epsilon\}$ is contained in U.

   Proof: For each $x \in K$ choose $r_x > 0$ such that the ball of radius $2r_x$ centered at x is inside U. Since the balls of radius r_x cover K, we can find a finite list of points $x_i \in K$ such that every $x \in K$ is within r_{x_i} of some x_i. Take ϵ to be the minimum of the r_{x_i}. For each $x \in K$ there is an x_i such that $d(x, x_i) < r_{x_i}$, so that for y satisfying $d(x,y) < \epsilon$ then $d(y, x_i) \le d(y,x) + d(x,x_i) < 2r_{x_i}$. Hence U contains the ϵ-ball centered at x.

B) A continuous function g defined on a compact metric space K with values in a metric space is uniformly continuous in the sense that for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(g(x), g(y)) < \epsilon$.

Proof: By the definition of continuity given $\epsilon > 0$ there exists $\delta_x > 0$ for each $x \in K$ such that $d(y,x) < \delta_x$ implies $d(g(y), g(x)) < \epsilon/2$. Cover K by open balls of radius $\delta_x/2$ and use compactness of K to find a finite list of points $x_i \in K$ such that for each $x \in K$ there is an x_i such that $d(x, x_i) < \delta_{x_i}/2$. Let δ be the minimum of δ_{x_i}. Let $x, y \in K, d(x,y) < \delta/2$, and $d(x, x_i) < \delta_{x_i}/2$. Then $d(y, x_i) \leq d(y,x) + d(x, x_i) < \delta \leq \delta_{x_i}$. Thus $d(g(y), g(x)) \leq d(g(y), g(x_i)) + d(g(x_i), g(x)) < 2\epsilon/2$. This gives the uniform continuity result.

To prove the proposition choose ϵ as in A) so that the ϵ-ball about each point in the compact set $g(K)$ is contained in Ω. Divide the unit square into equal sized sub-squares by vertical and horizontal lines such that points x, y in each sub-square satisfy $d(x,y) < \delta$ for the δ given by B) for the uniformly continuous H on the compact unit square. Then the image of each sub-square is contained in an open disk contained in Ω, so that we can apply Cauchy's theorem there. Since H is only assumed to be continuous the image of the boundary segments of sub-squares will only be continuous paths, so we need the following lemma to make sense of path integrals of holomorphic functions over continuous paths.

Lemma: Let $\alpha(t) : [0,1] \to \Omega$ be a continuous function and let $f(z)$ be holomorphic on Ω. Let ϵ be such that for each $t \in [0,1]$ all points within ϵ of $g(t)$ are contained in Ω. There exists a partition $[a_i, a_{i+1}]$ into subintervals such that $g([a_i, a_{i+1}])$ is contained in the ϵ-ball about $g(a_i)$ and in the ϵ-ball about $g(a_{i+1})$. Let L_i be the line segment joining $g(a_i)$ to $g(a_{i+1})$. Then $\sum \int_{L_i} f(z)dz$ is independent of the partition chosen, and if $\alpha(t)$ is piecewise smooth equals $\int_\alpha f(z)dz$.

Proof: Define the function $H(t,u) = \alpha(t)$ on the unit square. The discussion above shows that we can partition the square into sub-squares which map into open disks of radius ϵ such that the disk of twice the radius is contained in Ω. Take the partition of $[0,1]$ given by intersecting with the sub-squares. Since the image of $[a_i, a_{i+1}]$ is contained in the intersection of disks where Cauchy's theorem is true we will get the same answer for $\int_{L_i} f(z)\,dz$ as we get integrating on any piecewise smooth path joining $\alpha(a_i)$ to $\alpha(a_{i+1})$ which remains in the disk. Hence if we refine the partition by adding points between a_i, a_{i+1} we will not alter the value of the sum of integrals over line segments joining images of the partition points. Further if α is piecewise smooth on $[a_i, a_{i+1}]$ the path integral along α agrees with that along the line segment joining $\alpha(a_i), \alpha(a_{i+1})$ by Cauchy's theorem. This establishes the lemma.

If we define the integral of a holomorphic function on a continuous path as in the lemma using a suitable partition this extends the definition we used for piecewise smooth paths, and Cauchy's theorem remains true for such paths which lie in an open disk. Thus the subdivision argument works to show that the integral around the boundary of the image of the unit square of a holmorphic function is zero, establishing that integration of a holomorphic function on homotopic paths gives the same values.