Mathematics 503 Complex Analysis Fall 2017

Brief notes on homotopy of paths in the complex plane

Let $\alpha(t), \beta(t), t \in [0,1]$ be two continuous paths from z_0 to z_1 which lie in an connected open set Ω. We say that they are homotopic if there exists a continuous function $H(t,u) : [0,1] \times [0,1] \to \Omega$ such that $H(t,0) = \alpha(t), H(t,1) = \beta(t), H(0,u) = z_0, H(1,u) = z_1$. Intuitively, the homotopy deforms the path α to the path β, leaving the endpoints z_0, z_1 fixed.

We write $\alpha \sim \beta$ to indicate homotopy, which is an equivalence relation among paths. We will have need of the constant path 1_p at a point p, given by never moving from the start p, as well as the opposite path $\alpha_-(t) = \alpha(1-t)$. Paths can be composed when the end of the first is the start of the second, to give a path $\gamma\alpha$ parameterized by $\alpha(2t)$ for $t \in [0,1/2]$ and $\gamma(2t-1)$ for $t \in [1/2,1]$. Intuitively, we traverse α to it's end and then follow γ.

Homotopy respects equivalence classes and composition in the sense that $\alpha \sim \beta, \gamma \sim \eta$ implies $\gamma\alpha \sim \eta\beta$, as one sees immediately by composing the homotopy functions in the obvious way.

If α starts at z_0 and goes to z_1 then $\alpha_-\alpha \sim 1_{z_0}$ via the homotopy

$$H(t,u) = \begin{cases} \alpha(2t) & \text{if } 0 \le 2t \le (1-u) \\ \alpha_(1-u) & \text{if } (1-u) \le 2t \le u+1 \\ \alpha(2-2t) & \text{if } u+1 \le 2t \le 2 \end{cases}$$

while $\alpha\alpha_- = 1_{z_1}$. Intuitively we travel out the path α to $\alpha(1-u)$, rest there a bit, and then slide back to $\alpha(0)$.

We have $\alpha\alpha_- \sim 1_{\alpha(1)}, \alpha_-\alpha \sim 1_{\alpha(0)}, \alpha 1_{\alpha(0)} \sim \alpha, 1_{\alpha(1)}\alpha \sim \alpha$. This gives the set of homotopy classes of paths in Ω a structure that allows composition of certain paths, right and left identity elements under composition of homotopy classes and left and right inverses to a path up to homotopy.

Let α, β be two paths with initial point z_0 and terminal point z_1. Then $\alpha \sim \beta$ if and only if $\alpha\beta_- \sim 1_{z_1}$, so questions of homotopy of paths can be reduced to the study of closed paths.

We can relate the study of closed paths at points p to those at point q by letting η be a path from p to q. If α is a closed path at p, then $\eta\alpha\eta_-$ is a closed path at q. In particular if every closed path at q is homotopic to 1_q then $\eta\alpha\eta_- \sim 1_q$ so that $\alpha \sim \eta_- 1_q \eta \sim 1_p$ and hence every closed path at q is homotopic to a constant path at q.

Let Ω be star shaped with center z_0 and let α be a closed path at z_0. The homotopy $H(t,u) = uz_0 + (1-u)\alpha(t)$ has image in Ω since the set is star shaped, and gives $\alpha \sim 1_{z_0}$. By the discussion in the previous paragraphs we then have that any closed path at the point $p \in \Omega$ is homotopic to 1_p, so that any two paths in a star shaped set with the same initial and terminal points are homotopic.