**Mathematics 503 Complex Analysis Fall 2017**

## Calculus review

Recall that a function $F$ defined on an open subset $U$ of the real vector space $\mathbf{R}^n$ with values in $\mathbf{R}^m$ is differentiable at a point $x_0 \in U$ if there is a linear function $L(x)$ from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that

$$\lim_{x \to x_0} \frac{||F(x) - F(x_0) - L(x - x_0)||}{||x - x_0||} = 0$$

where $||y||$ is the usual Euclidean norm on $\mathbf{R}^m$.

A real valued function $F$ which is differentiable at $x_0$ has partial derivatives $F_{x_i}(x_0)$ since the limit may be taken over $x$ differing from $x_0$ only in position i.

The key to understanding the relation between existence of partial derivatives and differentiability is the following 1-variable calculus result:

Mean value theorem: Let f(x) be a real valued continuous function on an interval $[a, b]$ which is differentiable on $(a, b)$. Then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

The proof follows by recalling that a continuous function on a compact set takes on a maximum and a minimum and that if the derivative exists in the neighborhood of an extreme point it is 0 at the extreme point. Applying this to the function $g(x) = f(x) - s(x)$ where $s(x)$ is the function giving the secant line and noting $g(a) = g(b) = 0$ we see an extreme point occurs at some $c \in (a, b)$ and the derivative $g'(c) = 0$ which gives the result.

We now show that if a vector valued function on an open set $U \in \mathbf{R}^n$ has partial derivatives which are continuous in a neighborhood of $x_0 \in U$ then it is differentiable at $x_0$.

The essential idea already appears in the case of a real valued function of two variables. Let $u(x, y)$ be a real valued function on an open set $U$ in $\mathbf{R}^2$ which has both partial derivatives existing in a neighborhood of $(a, b) \in U$. Applying the mean value theorem in the first variable $u(a + h, b + k) - u(a, b + k) = u_x(x1, b + k)h = u_x(a, b)h + (u_x(x1, b + k) - u_x(a, b))h$ where $|x1 - a| < h$. Similarly using the mean value theorem in the second variable gives $u(a, b + k) - u(a, b) = u_y(a, b)k + (u_y(a, y1) - u_y(a, b))k$ where $|y1 - b| < k$.

Summing these expressions gives that $u(a + h, b + k) - u(a, b)$ equals

$$u_x(a, b)h + u_y(a, b)k + (u_x(x1, b + k) - u_x(a, b))h + (u_y(a, y1) - u_y(a, b))k$$

When the partial derivatives are continuous in a neighborhood of (a,b), this shows that $u(x, y)$ is differentiable at $(a, b)$. Under this assumption we have

$$u(a + h, b + k) - u(a, b) = u_x(a, b)h + u_y(a, b)k + |h + ik|\psi(h, k)$$

where

$$\psi(h,k) = ((u_x(x1, b+k) - u_x(a,b))h + (u_y(a, y1) - u_y(a,b))k)/|h+ik|$$

satisfies $\lim_{|h+ik|\to 0} |\psi(h,k)| = 0$ when $u_x, u_y$ are continuous near (a,b), verifying that continuous partial derivatives in a neighborhood of a point give the expression at the bottom of page 13 in Stein and Shakarchi and shows that $u(x,y)$ is differentiable.

A similar idea using the mean value theorem can be used to show that mixed partials of $u$ are equal at $(a,b)$ when $u_x, u_u, u_{xy}, u_{yx}$ are continuous in a neighborhood (Clairaut's theorem).

Consider the function $g(t) = u(t, b+k) - u(t,b)$. By the mean value theorem $g(a+h) - g(a) = u(a+h, b+k) - u(a+h, b) - (u(a, b+k) - u(a,b)) = g'(c_k)$ for some $c_k$ such that $|c_k - a| < |h|$. Hence

$$u(a+h, b+k) - u(a+h, b) - (u(a, b+k) - u(a,b)) = (u_x(c_k, b+k) - u_x(c_k, b))h.$$

If the derivative $u_{xy}$ of $u_x$ with respect to $y$ exists and is continuous near $(a,b)$ , the previous paragraphs show that the right hand side above is expressible as $(u_{xy}(c_k, b)k + k\psi(k))h$ with $|c_k - a| < |h|, \lim_{k\to 0}\psi(k) = 0$. Thus we have

$$g(a+t) - g(a) = t^2(u_{xy}(c_t, b) + \psi(t))$$

so that the limit as $t \to 0$ of $(1/t^2)(g(a+t) - g(a))$ is $u_{xy}(a,b)$ by continuity of $u_{xy}$.

The same logic applied to the function $g2(t) = u(a+h, t) - u(a,t)$ shows that $\lim(1/t^2)(g2(b+t) - g2(b)) = u_{yx}(a,b)$ when $u_{yx}$ is continuous near $(a,b)$.

Since $u(a+t, b+t) - u(a+t, b) - u(a, b+t) + u(a,b)$ is unchanged when the central two terms are permuted we see $g(a+t) - g(a) = g2(b+t) - g2(b)$. Ee obtain after dividing by $t^2$ and taking the limit that $u_{xy}(a,b) = u_{yx}(a,b)$.