1.

1.1. **Introduction.** A partial ordering on a set S is a relation $\leq$ between some elements of set which is reflexive ($s \leq s$), transitive ($s \leq t, t \leq u$ implies $s \leq u$), antisymmetric ($s \leq t$ and $t \leq s$ implies $s = t$).

For example, the set 2^X of all subsets of a set X is partially ordered by inclusion. The set of positive integers is partially ordered by divisibility.

A partial ordered set S in which any two elements are comparable is called a totally ordered set, for example the real line with the usual ordering, or the set of intervals $[0, t], t > 0$ ordered by inclusion or more generally a chain of sets each contained in another.

An element M of a partially ordered set S is called a maximal element if the only element greater than or equal to M is M itself. For any finite partially ordered set S there exists maximal elements, since one can start with any element of the set and compare it step by step with the other elements of the set, replacing the original choice each time a larger element is found.

An upper bound for a subset T of a partially ordered set S is an element $s_0 \in S$ such that for all$t \in T$ we have $t \leq s_0$.

The following result is often useful.

2.

2.1. **Zorn's Lemma.**

Theorem. *(Zorn's Lemma) Let S be a partially ordered set such that every totally ordered subset (= chain) has an upper bound. Then there exists a maximal element in S.*

Zorn's lemma is equivalent to other statements in set theory, for example the Axiom of Choice and the Well Ordering Principle. In this course we will work in an axiomatic set theory with these statements valid.

See https://gowers.wordpress.com/2008/08/12/how-to-use-zorns-lemma/ for tips on the use of Zorn's lemma in various aspects of mathematics.

3.

3.1. **Applications.**

Theorem. *Let V be a vector space over a field F. Let L be a linearly independent set of vectors in V. Let S be a spanning set of vectors. There exists a basis B of V containing L and contained in $L \cup S$.*

Proof. Let Σ be the set of all collections of vectors in V containing L and contained in $L \cup S$ and which form a linearly independent set. Σ is partially ordered by inclusion. If we take a chain $C \subset D \subset \cdots$ in Σ, the union of all sets in the chain is a set of vectors containing L and contained in $L \cup S$. It is linearly independent since a dependency relation $\sum c_i v_i = 0$ involves finitely many elements in the chain, hence takes place wholly in one of the chain elements. So every chain has an upper bound (the union of sets in the chain). By Zorn's lemma there exists a maximal element B of Σ. It is a set B of vectors containing L, which is linearly independent and contained in $L \cup S$. It spans V, since if not some $s \in S$ is not in B, so $B \cup \{s\}$ is in Σm contradicting maximality. □

Corollary. *Let S be a spanning set of a vector space V. There is a basis of V consisting of vectors in S. Any linearly independent subset of V extends to a basis of V.*

Proof. Take L in the theorem to be the set consisting of some nonzero vector in S, so there exists a basis B contained in S. For the last statement, given a linearly independent set L use the theorem with S taken to be all vectors of V.

□

Proposition. *If S spans a vector space V and L is a finite linearly independent set then there is a spanning set S' bijective to S such that L is a subset of S'. Hence the cardinality of S is at least that of L.*

Proof. Let the elements of L be $w_i, 1 \leq i \leq n$. w_1 has an expression $w_1 = c_1 s_1 + ...$ with $c_1 \neq 0$. Replace s_1 in S by w_1 to get S_1 spanning and containing w_1. The span is unchanged, and S is bijective to S_1. Suppose we have constructed a spanning set S_j bijective to S containing the first j elements of L. If w_{j+1} is in S_j let $S_j = S_{j+1}$. If not, write w_{j+1} as a sum of elements t_k of S_j. By linear independence some t_k not in L must appear with nonzero coefficient. Replace t_k by w_{j+1} to get S_{j+1}. Then t_k is in span of S_{j+1}. So S_{j+1} is bijective with S_j and contains $w_1, ..., w_{j+1}$. Thus by induction, there is a spanning set S' containing L and bijective to S. □

Corollary. *If V has a finite spanning set, then all bases have the same number of elements, the dimension.*

Proof. There is a finite basis $b_1, \ldots, b_n$ which is a subset of the finite spanning set. Given any finite set with more than n vectors $v_1, ldots, v_k$ the proposition above shows that they are dependent. Thus every linearly independent set has cardinality at most that of a spanning set, so the cardinality of one basis does not exceed that of another. □