Top Ten Jordan Normal Form Applications - Algebra 452 - Spring 2018

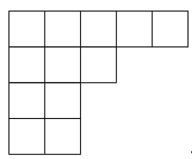
There are many applications of the Jordan Normal Form for complex square matrices A. Jordan normal form guarantees that A is similar to a block matrix with each block of the form $(\lambda I_k + N)$ for some eigenvalue λ of A, some identity matrix I_k and a $k \times k$ matrix N with zeros everywhere except right below the diagonal, where 1 appears. Note that $N^k = 0$ while N^{k-1} is nonzero.

1. Two $n \times n$ complex matrices A, B have the same Jordan form if and only if

$$\dim\ker(tI - A)^k = \dim\ker(tI - B)^k$$

   for all complex numbers t and all $k = 1, 2, 3, \ldots, n$. For a given λ, the geometric multiplicity $\dim\ker(\lambda I - A)$ (which is the number of blocks with diagonal λ) is at most the algebraic multiplicity of λ as a root of the characteristic polynomial of A (which is the sum of the heights of the blocks with diagonal λ).

2. The information in an $n \times n$ matrix A of Jordan blocks with the same number λ on the diagonal can be schematically represented by a tableau of boxes by arranging the blocks in decreasing size and making the columns in the tableau consist of a column of boxes the same height as the corresponding Jordan block. For example, if the Jordan blocks are square matrices of size 4,4,2,1,1 we obtain the tableau



   The numbers $r_k = \dim\ker(\lambda I - A)^k$ are the number of boxes lying in or above row k of the tableau. For the tableau above these are $r_1 = 5, r_2 = 8, r_3 = 10, r_4 = 12$.

3. Let $\lambda_1, \ldots, \lambda_j$ be distinct complex numbers. The number of similarity classes of complex matrices with characteristic polynomial $\prod_{i=1}^{j}(t-\lambda_i)^{e_i}$ is the number of collections of tableaux $T_1, \ldots, T_j$ as above with e_i boxes in tableau T_i. This is the same as the number of isomorphism classes of Abelian groups of order $p_1^{e_1} \cdots p_j^{e_j}$ where p_i are distinct primes. There exist uniform methods to prove Jordan form and the fundamental theorem on finite Abelian groups as consequences of a general principle.

4. Let $\lambda_1, \ldots, \lambda_j$ be distinct complex numbers. The number of similarity classes of complex $n \times n$ matrices with $\prod_{i=1}^{j}(A - \lambda_i)^{e_i} = 0$ is the number of collections of tableaux $T_1, \ldots, T_j$ as above with no more than e_i boxes in a column of tableau T_i, and n boxes in all tableaux T_j taken together. This is the same as the number of isomorphism classes of Abelian groups H of order $p_1^{r_1} \cdots p_j^{r_j}$ where $r_1 + \cdots + r_j = n$ such that for all $h \in H$, $h^e = 1$ for $e = p_1^{e_1} \cdots p_j^{e_j}$. In particular, given eight 6×6 complex matrices with cube 0, at least two are similar, by investigating tableaux with 6 boxes and at most 3 rows.

5. Powers of an $m \times m$ Jordan block of form $\lambda I + N$ are easy to compute by the binomial theorem

$$(\lambda I+N)^k = \lambda^k I + k\lambda^{k-1}N + \frac{k(k-1)}{2}\lambda^{k-2}N^2 + \ldots + \frac{k(k-1)\cdots(k-m+1)}{m!}\lambda^{k-m}N^m$$

   Note that the entries just below the diagonal in this matrix will all equal $k\lambda^{k-1}$. This expression for powers is valid for fractional k as well as integer, and is a polynomial in N since some power of N is the zero matrix. Hence every invertible square complex matrix is an m-th power for $m = 2, 3, \ldots$.

6. If A is a square complex matrix such that some positive power is the identity, then A is diagonalizable, since 5. above shows that a Jordan block of size larger than 1×1 can not have a positive power equal to the identity, since it would have nonzero entries just below the diagonal.

7. The matrix $\lim_{k\to\infty} A^k$ exists if and only if the eigenvalues of A except 1 are in the interior of the unit circle in the complex plane and the algebraic multiplicity of 1 is equal to $\dim\ker(A-I)$. This follows from the fact that the limit exists if and only if it exists for each block of the Jordan form of A. By 5. above the limit exists for a Jordan block of size m with eigenvalue λ if and only if the limit as $k \to \infty$ of

$$\lambda^k I + k\lambda^{k-1}N + \frac{k(k-1)}{2}\lambda^{k-2}N^2 + \ldots \lambda^{k-m}\frac{k(k-1)(k-2)\cdots(k-m+1)}{m!}N^m$$

   exists which is true if and only if $|\lambda| < 1$ or $\lambda = 1$ and $m = 1$.

8. The usual series $\exp(z) = \sum z^n/n!$ makes sense to define a matrix $\exp(N)$ as a polynomial in N when some power of N is 0. Then $\exp(\lambda I + N) = e^\lambda \exp(N)$ where each side is written as a polynomial in N with complex number coefficients. Since any complex matrix is similar to a Jordan form, the series for $\exp(A)$ converges for any square complex matrix. Further $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute. The Taylor series $\log(1-z) = -(z + z^2/2 + z^3/3 + \cdots)$ can be used to define the logarithm of an invertible Jordan block matrix, so that every invertible matrix is of the form $\exp(B)$ for some matrix B.

9. For any element y_0 in $\mathbf{C}^n$ the n-tuple of functions $y = \exp(Az)y_0$ is the unique solution to the matrix differential equation $y'(z) = Ay(z), y(0) = y_0$. Since every constant coefficient n-th order initial value problem can be recast as a first order matrix differential equation, this explains why repeated roots to the characteristic equation can lead to solutions involving terms of form $z^j e^{cz}$.

10. A square matrix is similar to its transpose, since this is true for Jordan blocks by listing the basis vectors in a Jordan string in reverse order.