Mathematics 435 **Geometry** **Fall 2021**

Conics

The Greeks introduced the term conic section to describe the figures formed by intersecting a cone with a plane. If we take the cone to be given by $x^2 + y^2 - z = 0$ and the plane by $\alpha x + \beta y + \gamma z = 0$ and describe points on the plane as linerar combinations of vectors we cab describe the points on the conic as

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

(where x,y are coordiantes on the plane) which we will call an affine conic.

When we homogenize the preceeding equation so all terms have total degree 2 we obtain the equation of a projective conic in the projective plane $\mathbf{P}^2(\mathbf{R})$ as the set of homogeneous coordinates $[x, y, z]$ satisfying

$$ax^2 + 2bxy + cy^2 + 2dxy + 2eyz + fz^2 = 0$$

In each case we can rewrite the equations of conics using symmetric matrices and use linear algebra to understand them.

For a general projective conic consider the symmetric 3×3 matrix

$$S = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

Then the transpose S^t is the same as S and the equation of the conic can be written

$$(x, y, z)S \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Letting $v = (x, y, z)^t$ be the column vector of the homogeneous coordinate of a point in the projective plane the equation can be written as

$$v^t S v = 0.$$

We need the following result about symmetric real matrices.

Theorem: If S is a symmetric $n \times n$ real matrix then all its eigenvalues are real and there is a basis of $\mathbf{R}^n$ formed by an orthonormal set of eigenvectors of S. The matrix P with columns given by these eigenvectors satisfies that PP^t is the identity matrix, and $P^t SP$ is a diagonal matrix with the eigenvalues of the matrix on the diagonal (in corresponding order to the eigenvectors) .

Proof:(for $n <= 3$). Every 1×1 matrix is symmetric and diagonal. When $n = 2$ the matrix is

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

which has eigenvalues the roots of $\lambda^2 - (a+c)\lambda + (ac - b^2)$ which by the quadratic formula are given by $\lambda = (a+c) \pm \sqrt{((a+c)^2 - 4(ac-b^2)}$. Since the term under the square root equals $(a-c)^2 + 4b^2$ is not negative, all roots are real, so there is a real eigenvalue λ_1 and corresponding real eigenvector v_1. Since S is symmetric if $w^t v_1 = 0$, then $(Sw)^t v_1 = 0$ so that the orthogonal complement to v_1 is an eigenspace for S as well. Take the columns of P to be the normalized column vectors $v_1/|v_1|, v_2/|v_2|$ and $P^t SP$ is a diagonal matrix with the eigenvalues of S on the diagonal.

For n=3, we can use that every monic cubic polynomial has at least one real root by the intermediate value theorem, so that S has a real eigenvalue λ_1 and corresponding real eigenvector v_1. The orthogonal complement to v_1 is a two dimensional subspace W and multiplication by S maps W to itself. By the case n=2 of the previous paragraph , we can choose an orthonormal basis of W formed by eigenvectors of S. Together with $v_1/|v_1$ we have an orthonormal basis of $\mathbf{R}^3$ of eigenvectors of S, and the matrix P with these vectors as columns satisfy $P^t SP$ is a diagonal matrix with the eigenvalues of S down the diagonal.

Multiplication by P^t induces a map of the projective plane to itself, and since P is an orthogonal matrix $v = PP^t v$ so that the image points $P^t v$ satisfy

$$0 = v^t Sv = v^t PP^t SPP^t v = (P^t v)^t (P^t SP)(P^t v)$$

so that the equation of the image is given by

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.$$

If the product of eigenvalues is nonzero we can multiply by a diagonal matrix with diagonal entries $1/\sqrt{|\lambda_i|}$ to conclude that after applying a projective transformation the equation of the conic may be written as

$$x^2 + y^2 - z^2 = 0.$$

Note that when the eigenvalues are nonzero with the same signs the equation becomes $x^2 + y^2 + z^2 = 0$ which is not satisfied by any point in the projective plane.

If the product of the eigenvalues is zero then the equation is product of linear polynomials in x, y, z so the zero set degenerates to conics containing lines. The list of equations obtained becomes

$$x^2 + y^2 - z^2 = 0, x^2 + y^2 = 0, x^2 - y^2 = 0, x^2 = 0$$

Up to projective transformation, all conics can be mapped by a projective transformation to one of the above list. The rank of the symmetric matrix describes how many variables appear in the resulting equation. In particular, any conic that does not contain a line is projectively equivalent to the circle $x^2 + y^2 - z^2 = 0$ in the real projective plane.

For affine conics, we can write the equation as

$$[x, y]M \begin{pmatrix} x \\ y \end{pmatrix} + 2[d, e] \begin{pmatrix} x \\ y \end{pmatrix} + f = 0$$

where the symmetric matrix $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We can find an orthogonal 2×2 matrix so that $Q^t M Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and then the linear map obtained by multiplication by Q^t maps the conic to one with equation

$$\lambda_1 x^2 + \lambda_2 y^2 + 2[d, e]Q \begin{pmatrix} x \\ y \end{pmatrix} + f = 0.$$

If $\lambda_1 \lambda_2 \neq 0$ we can eliminate the linear terms in x, y by completing the square (which replaces (x,y) by a translate)and obtaining an equation of the form

$$\lambda_1 x^2 + \lambda_2 y^2 + g = 0$$

which is an ellipse or hyperbola according to determinant of M positive or negative.

If $\lambda_1 \neq 0, \lambda_2 = 0$ then translating x, y will yield an equation of form

$$x^2 + \alpha y = 0$$

which is a parabola if $\alpha \neq 0$ and a line otherwise.

Examples : These are taken from section 10.7 of the book "Geometry by its History". See the course web page for a link to this supplemental text. Graphs of the conics are drawn there.

The affine conic

$$36x^2 - 24xy + 29y^2 + 120x - 290y + 545 = 0$$

has matrix $M = \begin{pmatrix} 36 & -12 \\ -12 & 29 \end{pmatrix}$ with eigenvalues $\lambda_1 = 20, \lambda_2 = 45$. Since $det(M) > 0$ this is an ellipse in the Euclidean plane. The corresponding eigenvectors are $\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$ so the conic is a translate by [0,5] of a rotation of a standard ellipse so the axes line up with these vectors. The new equation is $20x^2 + 45y^2 = 180$ which has its long axis horizontal.

A second example (10.57) in that text is the equation

$$16^2 - 24xy + 9y^2 - 130x - 90y + 50 = 0.$$

The matrix matrix $M = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}$ has determinant zero, so this is a parabola. The eigenvalues $25, 0$ correspond to the same eigenvectors as in the previous example, so the graph is obtained by translating a rotation of a standard parabola