Mathematics 435 **Geometry** **Fall 2019**

REVIEW QUESTIONS for FINAL EXAMINATION

The questions below cover material covered in Mathematics 435 this semester. The emphasis is on aspects discussed in weeks 9 through 14 on the syllabus, since the review problems for exams 1 and 2 cover earlier weeks. The final exam covers the whole semester, giving an opportunity to unify the various geometries covered.

1. Let S^2 be the unit sphere $\{x^2 + y^2 + z^2 = 1\}$ in three space.
   a) Consider rotation R_1 by $\pi/4$ about the x-axis sending $(x, y, z) \mapsto (x', y', z')$. Find a unit quaternion q so that $q(xi + yj + zk)q^{-1} = (x'i + y'j + z'k)$
   b) Let the rotation R_2 be rotation by $\pi/2$ about the y-axis. Show that $R1 \circ R2$ is a rotation and describe it explicitly by giving its axis and degree of rotation.

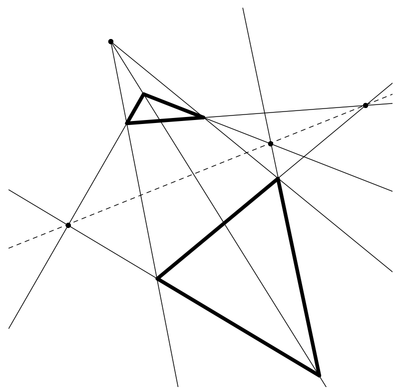
2. Consider the following points on the unit sphere

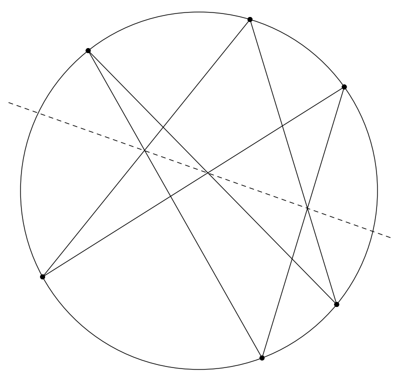
$$P = \frac{(-1, 0, 1)}{\sqrt{2}}, Q = (0, 0, 1), R = \frac{(-1, 1, 1)}{\sqrt{3}}$$

   a) Compute the angles in the triangle PQR.
   b) Compute the area of the triangle PQR.
   c) Which is the shortest side of the triangle, and which is the longest? Explain your answer by computing the lengths of all sides.
   d) Suppose the sphere is rotated around the y-axis by $\pi/2$ radians counter clockwise (looking from the positive y-axis towards 0). Find the coordinates of the vertices of the rotated spherical triangle.

   Hint: Draw the points on an orange or other sphere to get a feeling for the geometry.

3. The pictures on the left below illustrate various theorems we discussed this semester. Describe in words the statement of each theorem that is illustrated. If you wish, add labels to the diagram to reference in your description.





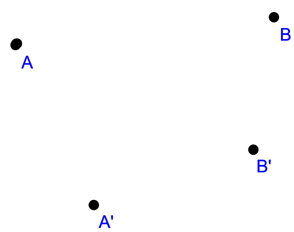
4. Show that in both Euclidean and non-Euclidean geometry the set of points equidistant from 2 distinct points is a line in that geometry (Hint: show first for two points with same second coordinate by reflecting in a vertical line,then reduce other cases to this.). Show that in each geometry any isometry is determined by its action on 3 non-collinear points, and is a product of at most 3 reflections.

5. Explain why reflections are the most important isometries in both Euclidean and non-Euclidean geometry.

6. Let $R_{P,\theta}$ be clockwise rotation of angle θ about the point P in the Euclidean plane.

a) What are the possible types for the isometry $R_{P,\theta} \circ R_{Q,\delta}$.

b) In the diagram below the point B' is the image of point B under a rotation R_1 about A. The point A' is the image of point A under a rotation R_2 about B. Show that $R_1 \circ R_2$ is a rotation about some point C. Sketch the point C in the picture and explain how you found it.



7. Prove the Pythagorean theorem in Euclidean geometry.

8. Recall that non-Euclidean lines in the upper half planes are vertical half-lines or upper halves of circles centered at a point on the real axis.

a) Let L_1, L_2 be non-Euclidean lines in the upper half plane. Show there is a Mobius transformation ϕ of the upper half plane satisfying $\phi(L_1) = L_2$.

b Let L_1, L_2 be as a above, and let $z_1 \in L1, z_2 \in L_2$ be points in the upper half plane on these lines. Show there is a Mobius transformation ϕ of the upper half plane satisfying $\phi(L_1) = L_2$ and $\phi(z_1) = z_2$.

9. Describe the non-Euclidean line L in the upper half plane which contains $-3+4i, 3+4i$ by an equation or geometric description. Compute the non-Euclidean distance between these two points. Find a non-Euclidean reflection which maps the non-Euclidean line L to the upper half of imaginary axis.

10. Give an explicit description of two non-Euclidean lines in the upper half plane through i and parallel to the non-Euclidean line $x = 3$.

11. The hyperbolic triangle bounded by the noneuclidean lines $x^2+y^2 = 1, x = -1/2, (x+1/2)^2 + y^2 = 1$ has vertices $-1/2 + \sqrt{3}/2i, -1/2 + i, -1/4 + \sqrt{15}/4i$. What is the area of the triangle?

12. Explain why two similar triangles in non-Euclidean or Spherical geometry must be congruent.

13. Show that set of points satisfying $-23x^2 + 72xy - 2y^2 + 3x - 7y = 25$ is a hyperbola in the Euclidean plane. What angle do the asymptotes of the hyperbola make with the x-axis?

14. State two theorems attributed to Thales that were studied in the course. Give a vector proof for each theorem.

15. Let $P = 4i, Q = i/8$ be points in the non-eulcidean upper half-plane. Which point is closer (in non-Eulidean distance) to the point i. Explain your answer.