Mathematics 251:08-10 Spring 2020

Students asked in the 2/11/20 lecture about one of the assigned written problems: 13.4 problem 21. This problem contains the hyperbolic trigonometric functions $\sinh(t) = (e^t - e^{-t})/2, \cosh(t) = (e^t + e^{-t})/2, \tanh(t) = \sinh(t)/\cosh(t)$ and $\mathrm{sech(t)} = 1/\cosh(\mathrm{t})$. Note that the derivative of sech(t) is $\cosh(t)$ and vice versa , and that $\cosh(t)^2 - \sinh(t)^2 = 1$. Finally the previous trigonometric identity can be divided by $\cosh(t)^2$ to obtain that $\mathrm{sech(t)}^2 + \tanh(\mathrm{t})^2 = 1$.

The book asks to show that the curvature is the hyperbolic secant function $1/\cosh(t)$. In fact we will see below that the correct answer for the curvature is the hyperbolic cosecant function $1/\sinh(t)$.

The problem considers the curve given parametrically by

$$\mathbf{r}(t) = (t - \tanh(t), \mathrm{sech(t)}).$$

To compute the curvature we need first to compute the tangent (or velocity) vector by differentiation to obtain

$$\mathbf{r}(t)' = (1 - \mathrm{sech(t)}^2, -\mathrm{sech(t)}\tanh(\mathrm{t})) = (\tanh(\mathrm{t})^2, -\mathrm{sech(t)}\tanh(\mathrm{t}))$$

The speed function is the length of this vector which equals $\tanh(t)$. Note that at $t = 0$ the speed equals 0, so the parameterization is not regular there. The curve has a sharp point at $t = 0$ and no tangent line at that point. This suggests that the curvature will not be defined there, but the book gives a curvature of 1 at $t = 0$ which is a hint that the answer to the problem is not what the book claims.

We can compute the curvature by computing the unit tangent vector and using the formula [2] of section 13.4 (since the speed is not constantly 1, this curve is not parameterized by arclength).

The unit tangent vector is

$$\mathbf{T}(t) = (\tanh(t), -\mathrm{sech(t)})$$

and has derivative

$$(1 - \tanh(t)^2, \mathrm{sech(t)}\tanh(\mathrm{t})) = \mathrm{sech(t)}(\mathrm{sech(t)}, \tanh(\mathrm{t}))$$

which has length sech(t)

The curvature is then

$$\kappa(t) = ||\mathbf{T}'(t)||/||\mathbf{r}(t)'|| = \mathrm{sech(t)}/\tanh(\mathrm{t}) = 1/\sinh(\mathrm{t}) = \mathrm{csch(t)}$$