Math 421
Summer 2019
Midterm exam 2
7/2/19

Name (Print): ____________________

This exam contains 4 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- You are allowed a formula sheet with Laplace transforms and Fourier series formulas only.

Problem	Points	Score
1	15	
2	15	
3	15	
4	20	
5	15	
6	20	
Total:	100	

1. (15 points) Solve

$$y'' + y = f(t), y(0) = y'(0) = 0$$

where

$$\begin{aligned} f(t) &= 1, 0 \leq t < 2 \\ &= e^t, t \geq 2. \end{aligned}$$

We have

$$\begin{aligned} f(t) &= 1 - \mathcal{U}(t-2) + e^t \mathcal{U}(t-2) \\ &= 1 - \mathcal{U}(t-2) + e^2 e^{t-2} \mathcal{U}(t-2). \end{aligned}$$

Thus

$$\begin{aligned} s^2 Y(s) + Y(s) &= \frac{1}{s} - \frac{e^{-2s}}{s} + e^2 \frac{e^{-2s}}{s-1} \\ Y(s) &= \frac{1}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)} + e^2 \frac{e^{-2s}}{(s-1)(s^2+1)}. \end{aligned}$$

We have

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1},$$

where $A = 1, Bi + C = \frac{1}{i} = -i$. Thus $B = -1, C = 0$. Also

$$\frac{1}{(s-1)(s^2+1)} = \frac{D}{s} + \frac{Es+F}{s^2+1},$$

where $E = \frac{1}{2}, Ei + F = \frac{1}{i-1} = -\frac{i+1}{2}$. Thus $E = F = -1/2$. Finally,

$$\begin{aligned} y(t) &= A + B\cos t + C\sin t - [A + B\cos(t-2) + C\sin(t-2)]\mathcal{U}(t-2) \\ &+ e^2 \big(e^{t-2} + E\cos(t-2) + F\sin(t-2)\big)\mathcal{U}(t-2). \end{aligned}$$

2. (15 points) Solve for $y(t)$ where

$$y(t) + \int_0^t (t-u) y(u) du = t.$$

We have

$$Y(s) + \frac{Y(s)}{s^2} = \frac{1}{s^2}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1}{s^2+1} \\ y(t) &= \sin(t). \end{aligned}$$

3. (15 points) Solve

$$y'' + y = \delta(t - 2\pi), y(0) = y'(0) = 0.$$

We have

$$\begin{aligned} s^2Y(s) + Y(s) &= e^{-2\pi s} \\ Y(s) &= \frac{e^{-2\pi s}}{s^2+1}. \end{aligned}$$

Thus

$$y(t) = \sin(t - 2\pi)\mathcal{U}(t - 2\pi).$$

4. Let

$$\begin{aligned} f(x) &= 1, 0 < x < 1 \\ &= -2, 1 \le x < 2. \end{aligned}$$

(a) (10 points) Find the Fourier sine series expansion of f. We have

$$f(x) = \sum_{i=1}^{n} b_n \sin(\frac{n\pi}{2}x)$$

where

$$\begin{aligned} b_n &= \frac{2}{2}\big[\int_0^1 \sin(\frac{n\pi}{2}x) - \int_1^2 2\sin(\frac{n\pi}{2}x)\big] \\ &= \frac{2}{n\pi}\big[-\cos(\frac{n\pi}{2}x)\Big|_0^1 + 2\cos(\frac{n\pi}{2}x)\Big|_1^2\big] \\ &= \frac{2}{n\pi}\big[1 - 3\cos(\frac{n\pi}{2}) + 2\cos(n\pi)\big]. \end{aligned}$$

(b) (10 points) Let the Fourier series in part a be $S(x)$. Find $S(1)$.
Since $f(x)$ is discontinuous at $x = 1, S(1) = \frac{1-2}{2} = \frac{-1}{2}$.

5. (15 points) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} y'' + \lambda y &= 0, 0 < x < L \\ y'(0) &= 0, y(L) = 0. \end{aligned}$$

Case 1: $\lambda = -\alpha^2 < 0$

We have $y(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$. The boundary conditions give

$$\begin{aligned} y'(0) = \alpha(c_1 - c_2) = 0 \\ y(L) = c_1e^{\alpha L} + c_2e^{-\alpha L} = 0. \end{aligned}$$

Solving for this system gives $c_1 = c_2 = 0$. Thus there is no negative eigenvalue.

Case 2: $\lambda = 0$ We have $y(x) = c_1 x + c_2$. The boundary conditions give

$$y'(0) = c_1 = 0$$
$$y(L) = c_2 = 0.$$

Thus 0 is not an eigenvalue.

Case 3: $\lambda = \alpha^2 > 0$ We have $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. The boundary conditions give

$$y'(0) = \alpha c_2 = 0$$
$$y(L) = c_1 \cos(\alpha L) = 0.$$

Thus we choose $\alpha = \frac{(2k+1)\pi}{L}, k = 0, 1, 2, \cdots$. The corresponding eigenvalues are

$$\lambda = \alpha^2 = \left(\frac{(2k+1)\pi}{L}\right)^2$$

and the corresponding eigenfunctions are

$$y(x) = \cos(\frac{(2k+1)\pi}{L} x).$$

6. (20 points) Solve the heat equation with Neumann boundary condition

$$\begin{aligned} u_t &= u_{xx}, 0 < x < \pi, t > 0 \\ u_x(t,0) &= u_x(t,\pi) = 0, \\ u(0,x) &= x. \end{aligned}$$

We have discussed that the solution has the form of the cosine series

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx),$$

where

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x dx = \pi \\ a_n &= \frac{2}{\pi} \int_0^\pi x \cos(nx) dx = -\frac{2}{n\pi} \int_0^\pi \sin(nx) dx = \frac{2}{n^2 \pi} \cos(nx)\Big|_0^\pi = \frac{2}{n^2\pi}((-1)^n - 1). \end{aligned}$$