Math 421
Summer 2019
Midterm exam 1
6/11/19

Name (Print): ____________________

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Find det A.

   Ans: Expand along the 4th column:

$$detA = -det \begin{bmatrix} 0 & 0 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = -3 \times (3-1) = -6.$$

2. (20 points) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$. Find A^{-1}, if it exists. If not, explains why.

We perform the following row operations:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 0 & -1 \\ 0 & -2 & 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 0 & -1 \\ 0 & 0 & -9 & 4 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4/9 & -1/9 & 2/9 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 5/9 & -1/9 & 2/9 \\ 0 & 1 & 0 & -2/9 & -5/9 & 1/9 \\ 0 & 0 & 1 & -4/9 & -1/9 & 2/9 \end{array}\right].$$

Therefore,

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 5 & -1 & 2 \\ -2 & -5 & 1 \\ -4 & -1 & 2 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. It is also given that the eigenvalues of A are -2,-2,4.

   (a) (10 points) Find all corresponding eigenvectors of A

$$\begin{aligned} \lambda &= -2, A - \lambda I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ \lambda &= 4, A - \lambda I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -41 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

   (b) (10 points) Since A is symmetric, it can be written in the form of $A = PDP^T$. Find P, D. We need to perform Gram-Schmidt on $\mathbf{v}_1, \mathbf{v}_2$. The projection of $\mathbf{v}_1$ on $\mathbf{v}_2$ is

$$\frac{\langle \mathbf{v}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

   Thus a new choice of eigenvector $\mathbf{v}_1'$ is

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}.$$

   Note how $\mathbf{v}_1'$ is an eigenvector and is orthogonal to $\mathbf{v}_2$. Finally we just need to normalize $\mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_3$ to obtain P and we have

$$\begin{aligned} P &= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \\ D &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \end{aligned}$$

4. Find the Laplace transform of the following functions:

   (a) (5 points) $f(t) = te^{2t}$.

$$F(s) = \int_0^\infty e^{-st}e^{2t}t = \int_0^\infty e^{-(s-2)t}t = \frac{1}{(s-2)^2}.$$

   (b) (5 points) $f(t) = (2t+1)^3$.

$$f(t) = 8t^3 + 12t^2 + 6t + 1.$$

Thus

$$F(s) = 8\frac{3!}{s^4} + 12\frac{2!}{s^3} + 6\frac{1}{s^2} + \frac{1}{s}.$$

   (c) (10 points)

$$\begin{aligned} f(t) &= t, 0 \leq t < 1 \\ &= 1, t \geq 1. \end{aligned}$$

$$\begin{aligned} F(s) &= \int_0^1 e^{-st}t\,dt + \int_1^\infty e^{-st}dt \\ &= -\frac{te^{-st}}{s}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt - \frac{e^{-st}}{s}\Big|_1^\infty \\ &= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s}-1) + \frac{e^{-s}}{s} \\ &= \frac{1}{s^2}(1-e^{-s}). \end{aligned}$$

5. Find the inverse Laplace transform of the following functions:

(a) (5 points) $F(s) = \frac{(s+1)^3}{s^4}$.

$$F(s) = \frac{s^3 + 3s^2 + 3s + 1}{s^4} = \frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}.$$

Thus

$$f(t) = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{3!}t^3.$$

(b) (10 points) $F(s) = \frac{s^2+1}{s(s-1)(s+1)(s-2)}$.

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2},$$

where

$$A = \frac{1}{2}, B = -1, C = -\frac{1}{3}, D = \frac{5}{6}.$$

Thus

$$f(t) = A + Be^t + Ce^{-t} + De^{2t}.$$

(c) (5 points) $F(s) = \frac{s}{(s^2+1)(s^2+4)}$.

$$F(s) = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}.$$

Thus

$$(As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) = s,$$

which is

$$(A + C)s^3 + (B + D)s^2 + (4A + C)s + 4B + D = s.$$

This gives

$$\begin{aligned} A + C &= 0 \\ 4A + C &= 1. \end{aligned}$$

Hence $A = \frac{1}{3}, C = -\frac{1}{3}$.

$$\begin{aligned} B + D &= 0 \\ 4B + D &= 0. \end{aligned}$$

Hence $B = D = 0$.

Finally,

$$f(t) = A\cos t + C\cos 2t.$$