Math 421
Summer 2019
Final exam
7/18/19

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- You are allowed a formula sheet with Laplace transforms, Fourier series formulas and Sturm-Liouville problems only.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
Total:	160	

1. (20 points) Solve

$$y'' + y = f(t), y(0) = y'(0) = 0$$

where

$$\begin{aligned} f(t) &= 1, 0 \leq t < 2 \\ &= 2, t \geq 2. \end{aligned}$$

We have

$$f(t) = 1 - \mathcal{U}(t-2) + 2\mathcal{U}(t-2) = 1 + \mathcal{U}(t-2).$$

Therefore

$$\begin{aligned} s^2Y(s) + Y(s) &= \frac{1}{s} + \frac{e^{-2s}}{s} \\ Y(s) &= \frac{1+e^{-2s}}{s(s^2+1)}. \end{aligned}$$

Now

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1},$$

where $A = 1, Bi + C = \frac{1}{i} = -i$. Thus $B = -1, C = 0$.
Finally,

$$y(t) = 1 - \cos t + (1 - \cos(t-2))\mathcal{U}(t-2).$$

2. (20 points) Solve for $y(t)$ where

$$y(t) + \int_0^t (t-u)y(u)du = 1.$$

We have

$$\begin{aligned} Y(s) + \frac{Y(s)}{s^2} &= \frac{1}{s} \\ Y(s) &= \frac{s}{s^2+1}. \end{aligned}$$

Thus $y(t) = \cos t$.

3. (20 points) Solve

$$y'' + y = \delta(t - \pi), y(0) = y'(0) = 0.$$

We have

$$\begin{aligned} s^2Y(s) + Y(s) &= e^{-\pi s} \\ Y(s) &= \frac{e^{-\pi s}}{s^2+1}. \end{aligned}$$

Thus $y(t) = \sin(t - \pi)\mathcal{U}(t - \pi)$.

4. (20 points) Solve the heat equation with non-homogeneous Dirichlet condition

$$\begin{aligned} u_t &= u_{xx}, 0 < x < \pi, t > 0 \\ u(t,0) &= 1, u(t,\pi) = 0, \\ u(0,x) &= 0. \end{aligned}$$

We have

$$u(t,x) = 1 - \frac{x}{\pi} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx).$$

$u(0,x) = 0$ gives

$$\sum_{n=1}^{\infty} a_n \sin(nx) = \frac{x}{\pi} - 1.$$

Thus

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} (\frac{x}{\pi} - 1) \sin(nx) dx \\ &= \frac{2}{\pi} (- \frac{\cos(n\pi)}{n} + \int_0^{\pi} \frac{\cos(nx)}{\pi n} dx + \frac{\cos(nx)}{n} \Big|_0^{\pi} dx) \\ &= -\frac{2}{n\pi}. \end{aligned}$$

5. (20 points) Solve the wave equation with homogeneous Neumann condition

$$\begin{aligned} u_{tt} &= 4u_{xx}, 0 < x < \pi, t > 0 \\ u_x(t,0) &= u_x(t,\pi) = 0, \\ u(0,x) &= 0 \\ u_t(0,x) &= 2. \end{aligned}$$

We have

$$u(t,x) = \frac{a_0}{2} + \frac{b_0}{2} t + \sum_{n=1}^{\infty} a_n \cos(2nt)\cos(nx) + \sum_{n=1}^{\infty} b_n \sin(2nt)\cos(nx).$$

$u(0,x) = 0$ gives $a_n = 0, n = 0, 1, 2, \cdots$. $u_t(0,x) = 2$ gives

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} 2nb_n \cos(nx) = 2.$$

Thus

$$\begin{aligned} b_0 &= \frac{2}{\pi}\int_0^{\pi} 2dx = 4 \\ b_n &= \frac{2}{\pi}\int_0^{\pi} 2\cos(nx)dx = 0. \end{aligned}$$

Thus $u(t,x) = 2t$.

6. (20 points) Solve the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, 0 < x < \pi, 0 < y < 2\pi \\ u(0,y) &= u(\pi,y) = 0, \\ u(x,0) &= u(x,2\pi) = 1. \end{aligned}$$

We first solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, 0 < x < \pi, 0 < y < 2\pi \\ u(0,y) &= u(\pi,y) = 0, \\ u(x,0) &= 0, u(x,2\pi) = 1. \end{aligned}$$

The homogeneous Dirichlet direction is the x direction, thus the solution has the form

$$u_1(t,x) = \sum_{n=1}^{\infty} a_n \cosh(ny)\sin(nx) + \sum_{n=1}^{\infty} b_n \sinh(ny)\sin(nx).$$

$u_1(x,0) = 0$ gives $a_n = 0, \forall n$. $u_1(x,2\pi) = 1$ gives

$$\sum_{n=1}^{\infty} b_n \sinh(2n\pi)\sin(nx) = 1.$$

Therefore

$$\begin{aligned} \sinh(2n\pi)b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = \frac{2}{n\pi}(1 - \cos(n\pi)) \\ b_n &= \frac{2}{n\pi\sinh(2n\pi)}(1 - \cos(n\pi)). \end{aligned}$$

We now solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, 0 < x < \pi, 0 < y < 2\pi \\ u(0,y) &= u(\pi,y) = 0, \\ u(x,0) &= 1, u(x,2\pi) = 0. \end{aligned}$$

The homogeneous Dirichlet direction is also the x direction, thus the solution also has the form

$$u_2(t,x) = \sum_{n=1}^{\infty} c_n \cosh(ny)\sin(nx) + \sum_{n=1}^{\infty} d_n \sinh(ny)\sin(nx).$$

$u_2(x,0) = 1$ gives

$$\sum_{n=1}^{\infty} c_n \sin(nx) = 1.$$

Thus $c_n = \frac{2}{\pi}\int_0^\pi \sin(nx)dx = \frac{2}{n\pi}(1-\cos(n\pi))$. Finally $u_2(x,2\pi) = 0$ gives

$$\cosh(2n\pi)c_n + \sinh(2n\pi)d_n = 0.$$

Thus $d_n = -\coth(2n\pi)a_n$.

The solution to the original problem is $u(t,x) = u_1(t,x) + u_2(t,x)$.

7. (20 points) Solve the heat equation with mixed homogeneous boundary condition

$$\begin{aligned} u_t &= 5u_{xx}, \\ u(0,x) &= 1, \\ u(t,0) &= 0, u_x(t,2\pi) = 0. \end{aligned}$$

The associated Sturm Lioville problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) = y'(2\pi) &= 0 \end{aligned}$$

has eigenvalues $\lambda_n = \left(\frac{2n+1}{4}\right)^2$ and eigenfunctions $y_n(x) = \sin(\frac{2n+1}{4}x)$. Therefore the solution has the form

$$u(t,x) = \sum_{i=1}^{n} a_n e^{-\left(\frac{2n+1}{4}\right)^2 5t} \sin(\frac{2n+1}{4}x).$$

$u(0,x) = 1$ gives

$$\sum_{i=1}^{n} a_n \sin(\frac{2n+1}{4}x) = 1.$$

Thus

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} \sin(\frac{2n+1}{4}x)dx = \frac{4}{(2n+1)\pi}.$$

8. (20 points) Solve the wave equation with non-homogeneous Dirichlet condition

$$\begin{aligned} u_{tt} &= 9u_{xx}, 0 < x < \pi, t > 0 \\ u(t,0) &= 0, u(t,\pi) = 1, \\ u(0,x) &= 0 \\ u_t(0,x) &= 1. \end{aligned}$$

We have

$$u(t,x) = \frac{x}{\pi} + \sum_{n=1}^{\infty} a_n \cos(3nt)\sin(nx) + \sum_{n=1}^{\infty} b_n \sin(3nt)\sin(nx).$$

$u(0,x) = 0$ gives

$$\sum_{n=1}^{\infty} a_n \sin(nx) = -\frac{x}{\pi}.$$

Thus

$$\begin{aligned} a_n &= -\frac{2}{\pi}\int_0^{\pi} \frac{x}{\pi}\sin(nx)dx \\ &= \frac{2}{n\pi}\cos(n\pi). \end{aligned}$$

$u_t(0,x) = 1$ gives

$$\sum_{n=1}^{\infty} 3nb_n \sin(nx) = 1.$$

Thus

$$\begin{aligned} 3nb_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = \frac{2}{n\pi}(1-\cos(n\pi)) \\ b_n &= \frac{2}{3n^2\pi}(1-\cos(n\pi)). \end{aligned}$$