Let X and Y be two random variables both normally distributed with mean zero, variance one and correlation coefficient ρ.

1. Find the density and the distribution function of the random variable $Z \triangleq \frac{X}{Y}$. *Hint: Use the first question in the previous problem.*

2. Compute the probability $\mathbb{P}(X < 0, Y > 0)$.

Solution

1: We can represent

$$\begin{aligned} X &= Z_1 \\ Y &= \rho Z_1 + \sqrt{1-\rho^2} Z_2, \end{aligned}$$

where Z_1, Z_2 are independent standard Normals. This is the Cholesky decomposition.

Then (since the distribution of Y/X and X/Y are the same by symmetry)

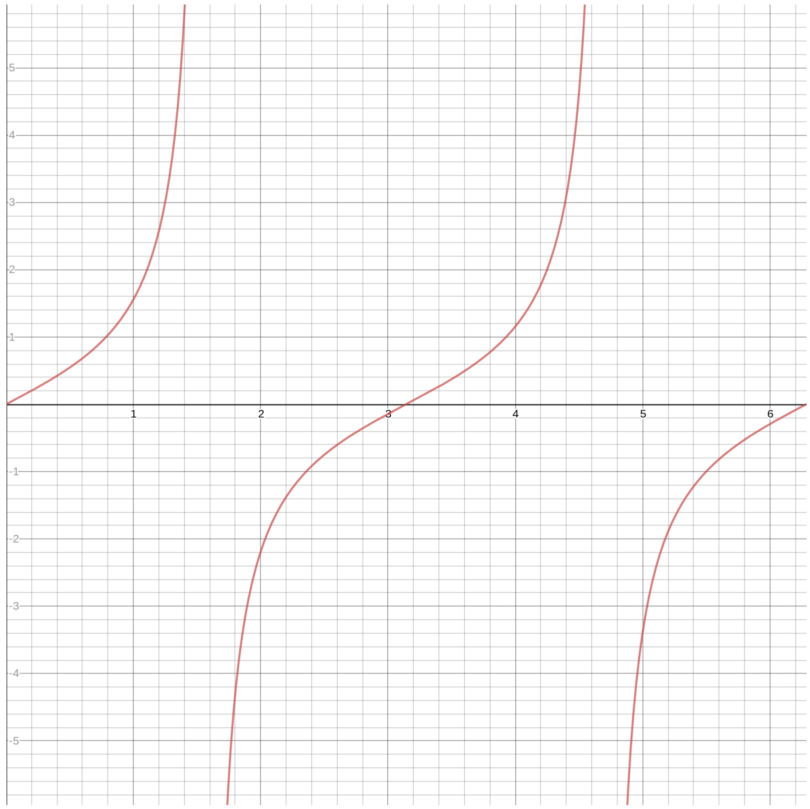
$$\frac{Y}{X} = \rho + \sqrt{1-\rho^2}\frac{Z_2}{Z_1}.$$

By inspecting the joint distribution of Z_1, Z_2 in terms of polar coordinate, we can see that $\frac{Z_2}{Z_1}$ has a distribution that corresponds to $\tan(U[0, 2\pi])$. This also determines the distribution of $\frac{Y}{X}$.

2: By symmetry,

$$\begin{aligned} P(Y > 0, X < 0) &= \frac{1}{2} P(\frac{Y}{X} < 0) \\ &= \frac{1}{2} P(\rho + \sqrt{1-\rho^2}\frac{Z_2}{Z_1} < 0) \\ &= \frac{1}{2} P(\frac{Z_2}{Z_1} < -\frac{\rho}{\sqrt{1-\rho^2}}). \end{aligned}$$

Consider the following graph of $y = \tan x$ on $[0, 2\pi]$ (since $\frac{Z_2}{Z_1}$ has a distribution that corresponds to $\tan(U[0, 2\pi])$)



We observe two things:

a) The region $x : \tan(x) < m$ depends on whether $m > 0$ or $m < 0$

b) The region $x : \tan(x) < m$ does not correspond directly to $x < \tan^{-1}(m)$ since $\tan^{-1}(m)$ maps to the interval $[-\pi/2, \pi/2]$.

Thus if $\rho \leq 0$:

$$\begin{aligned}
\frac{1}{2}P(\frac{Z_2}{Z_1} < -\frac{\rho}{\sqrt{1-\rho^2}}) &= \frac{1}{2}P\Big(U[0, 2\pi] \in [0, -\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}] \\
&\cup(\pi/2, -\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}} + \pi) \cup (3\pi/2, 2\pi)\Big) \\
&= \frac{1}{4\pi}(\pi - 2\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}) \\
&= \frac{1}{4} - \frac{1}{2\pi}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}.
\end{aligned}$$

Some reality check : If $\rho = -1$ then $P(X < 0, Y > 0) = P(Y > 0) = 1/2$. This is also the result we obtain by plugging $\rho = -1$ into the above expression. If $\rho = 0$ then $P(X < 0, Y > 0) = 1/4$, again agreeing with what we obtain when we plug in the formula.

If $\rho > 0$:

$$\begin{aligned}
\frac{1}{2}P(\frac{Z_2}{Z_1} < -\frac{\rho}{\sqrt{1-\rho^2}}) &= \frac{1}{2}P\Big(U[0,2\pi] \in [\pi/2, \pi - \tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}] \\
&\quad \cup (3\pi/2, -\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}} + 2\pi)\Big) \\
&= \frac{1}{4\pi}(\pi - 2\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}) \\
&= \frac{1}{4} - \frac{1}{2\pi}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}},
\end{aligned}$$

which is the same answer as we obtained before. Thus in either case,

$$P(Y > 0, X < 0) = \frac{1}{4} - \frac{1}{2\pi}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}.$$