Math 478
Spring 2019
Final exam
5/10/19

Name (Print): ____________________________

This exam contains 6 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. To earn extra income in the upcoming summer, Professor T. opens a hot dog stand in front of Scott Hall that starts daily at 8 AM. From 8 until 10 AM. customers (all are students from Rutgers) visit the stand, one at a time, on average, at a steadily increasing rate that starts with an initial rate of 2 customers per hour at 8 AM and reaching 5 customers per hour at 10 AM. From 10 AM to 2 PM the rate remains constant at 5 customers per hour. From 2 PM until the closing time at 5 PM, the rate drops steadily until it reaches 1 customer per hour at 5 PM.

   (a) (10 points) Find the probability that more than 5 customers visit the stand from 9 AM to 12 PM.

   We have $\int_9^{12} \lambda(t)dt = (3.5+5)/2 + 5 \times 2 = 14.25$. The probability is

$$1 - P(\text{ Poisson } (14.25) \leq 5) = 1 - \sum_{k=0}^{5} e^{-14.25} \frac{14.25^k}{k!}.$$

   (b) (10 points) Suppose that the amount of hot dogs each student buys is independently distributed according to a Poisson(2) distribution. Find the expected number of hot dogs sold from 9 AM to 12 PM.

   The number of hot dogs sold is a compound random vaiable: $S = \sum_{i=1}^{N} Y_i$, where Y_i's are iid Poisson(2) and N is a Poisson(14.25). Thus $E(S) = E(N)E(Y_1) = 28.5$.

2. (20 points) Students from different campuses visit the hot dog stand according to the following distribution: 40% from College Ave, 20% from Busch, 30% from Livingston and 10% from Cook-Douglass. Find the expected amount of students professor T. has to serve before students from all campuses have visited his stand..

   This is the coupon problem with different success rates. The answer is

$$E(X) = \int_0^\infty \left[1 - (1-e^{-.4t})(1-e^{-.2t})(1-e^{-.3t})(1-e^{-.1t})\right] dt.$$

3. (20 points) At noon, students get in line for the hot dog stand. Professor T.'s serving time for each student is an Exp(.5) distribution (in minutes). Being impatient college students, the customers will only wait an Exp(.2) distribution and leave if they are not served by that time. When Tommy arrives at the hot dog stand, there were one person being served and 4 extra students waiting in line. What is the probability that Tommy will get served ?

   Ans: This is the server problem from the lecture notes with $n = 6$ and $\mu = 0.5, \theta = 0.2$. Thus the answer is $\frac{\mu}{n\theta+\mu} = \frac{.5}{1.7}$.

4. As the hot dog stand becomes more popular, professor T. decides to hire a graduate student who's also interested in stochastic modeling as an assistant. The student is supposed to start sharply at 8 AM each morning but he is not very consistent in this matter. In fact, he can arrive early (state 1), very early (state 2), on time (state 3), late (state 4), very late (state 5) at work depending on how he started the previous day. The conditional distribution of his time

showing up to work depending on the previous day can be captured by the following matrix

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.4 \\ 0.4 & 0 & 0.2 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0 & 0.2 & 0 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.4 \end{bmatrix}.$$

In the following, give your answers in terms of the power of some matrix Q that you specify.

(a) (10 points) On the first day to work, the student arrived late. What is the probability that he will not arrive on time in the next 15 work days?

$$Q = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.4 \\ 0.4 & 0 & 0.2 & 0.3 & 0.1 \\ 0 & 0 & 1 & 0 & 0 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.4 \end{bmatrix}.$$

The probability that he will be on time on some of the next 15 work days is Q^{15}_{43}. Thus the answer is $1 - Q^{15}_{43}$.

(b) (10 points) On the first day to work, the student arrived late. What is the probability that he will not arrive late or very late in the next 20 work days?

$$Q = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.6 \\ 0.4 & 0 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The probability that he will be late or very late on some of the next 20 work days is $\sum_{j=1}^{3} P_{4j}(1 - Q^{19}_{j3})$.

5. (20 points) With the new helpful assistant, now two students can be served simultaneously at the stand. Suppose that professor T.'s serving time is Exp(.5) and the assistant's serving time is Exp(1) minutes. Again students are still impatient, so they will only wait an Exp(.2) minute distribution and leave if they are not served by that time. When Tomy arrives at the hot dog stand, there were two person being served and 5 extra students waiting in line. What is the probability that Tommy will get served ?

   We can view the two servers serving simultaneously as one single server with serving time equalling the minimum of the two, which is an Exp(1.5) random variable. Again this will be the server problem in the lecture with $n = 5, \mu = 1.5, \theta = 0.2, n = 7$ Thus the probability is $\frac{\mu}{n\theta + \mu} = \frac{1.5}{2.9}$.

6. After his success with the hot dog stand, professor T. puts all of his money into investment by buying the S&P 500 index. The future distribution of this index can be described as:

$$S_t = 3000e^{0.05W_t},$$

where t is in years and W_t is a Brownian motion. Professor T. will retire to the Bahamas if the index reaches 5000 before it reaches 2000. He will return to selling hot dogs if the reverse happens.

(a) (10 points) What is the probability that professor T. will end up retiring in the Bahamas instead of selling hot dogs?

The probability is the same as the probability that the Brownian motion will hit $A = 20\ln\frac{5}{3}$ before it hits $-B = 20\ln\frac{2}{3}$. Thus the answer is $\frac{B}{A+B}$.

(b) (10 points) What is the probability that the S&P 500 will reach 5000 before 5 years?

This is the same as the probability that $T_A \le 5$. Thus it is the same as $2P(W_5 \ge 20\ln\frac{5}{3}) = 2P(Z \ge \frac{20}{\sqrt{5}}\ln\frac{5}{3})$.

7. (20 points) In his dream scenario of retiring to the Bahamas, professor T. will own five yatchs. Each month, he chooses a yatch randomly with probability $p_i, i = 1, \cdots, 5$ to tour the islands. The service fee (while touring) of the ith yatch is $U[0, i10^5]$ dollars. Find the expectation and variance of professor T.'s monthly service fee for the yatchs.

Let Y be the yatch chosen for the month. Then $Y = i$ with probability $p_i, i = 1, \cdots, 5$. Let X be the monthly service fee. Then $X|Y = U[0, Y10^5]$. Thus $E(X|Y) = \frac{Y}{2}10^5$. Thus $E(X) = E(E(X|Y)) = \sum_{i=1}^{5} \frac{ip_i}{2}10^5$. On other hand, $Var(X|Y) = \frac{Y^2}{12}10^{10}$. And thus

$$E(Var(X|Y)) = \sum_{i=1}^{5} \frac{i^2 p_i}{12}10^{10}.$$

On the other hand,

$$Var(E(X|Y)) = \frac{10^{10}}{4}Var(Y) = \frac{10^{10}}{4}(\sum_{i=1}^{5} i^2 p_i - (\sum_{i=1}^{5} ip_i)^2).$$

Thus

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y)) = \sum_{i=1}^{5} \frac{i^2 p_i}{12}10^{10} + \frac{10^{10}}{4}(\sum_{i=1}^{5} i^2 p_i - (\sum_{i=1}^{5} ip_i)^2).$$

8. (20 points) Also while in the Bahamas, professor T. will purchase a private jet. There are 4 models available, the $G_i, i = 5, \cdots, 8$. Professor T. can only afford one private jet so he will choose a model G_i randomly with probability $p_i, i = 5, \cdots 8$. Suppose each model G_i has a life time distributed as $\mathrm{Exp}(\lambda_i)$ in years. Let X be the life time of professor T.'s private jet. Find the hazard rate of X.

This is the hazard rate problem from the lecture notes. Thus the answer is

$$r_X(t) = \frac{\sum_i p_i \lambda_i e^{-\lambda_i t}}{\sum_i p_i e^{-\lambda_i t}}.$$

9. (20 points) Let W_t be a Brownian motion. Perform the following calculations:

a. $E(W_t^2|W_s), s < t$.

$$\begin{aligned} E(W_t^2|W_s) &= E((W_t - W_s + W_s)^2|W_s) \\ &= E((W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2|W_s) \\ &= t - s + W_s^2. \end{aligned}$$

b. $E(W_t W_s W_r), r < s < t$.

This problem is from the homework and the answer is 0. The derivation is

$$\begin{aligned} E(W_t W_s W_r) &= E((W_t - W_s + W_s)W_s W_r) \\ &= E((W_t - W_s)W_s W_r) + E(W_s^2 W_r) \\ &= E(W_t - W_s)E(W_s W_r) + E(E(W_s^2 W_r|W_r)) \\ &= E(W_r(t - s) + W_r^3) = 0. \end{aligned}$$

c. $E(W_s|e^{W_t} = 5), s < t$.

$$E(W_s|e^{W_t} = 5) = E(W_s|W_t = \ln 5) = \frac{\ln 5s}{t},$$

from the Brownian bridge result.

d. $E(W_\tau^2)$ where $\tau \sim Exp(\lambda)$ distribution independent of W_t.

$$\begin{aligned} E(W_\tau^2) &= \int_0^\infty E(W_\tau^2|\tau = t)\lambda e^{-\lambda t} dt \\ &= \int_0^\infty E(W_t^2|\tau = t)\lambda e^{-\lambda t} dt \\ &= \int_0^\infty E(W_t^2)\lambda e^{-\lambda t} dt \\ &= \int_0^\infty t\lambda e^{-\lambda t} dt \\ &= \frac{1}{\lambda}. \end{aligned}$$

10. (20 points) Let W_t be a Brownian motion. Classify the following processes as stationary, weakly stationary or neither. Explain your reasoning.

a. $W_t^2, t \geq 0$.

Since $E(W_t^2) = t$ not a constant. This is neither.

b. $\frac{W_t}{\sqrt{t}}, t > 0$.

$E(\frac{W_t}{\sqrt{t}}) = 0$. On the other hand,

$$Cov(\frac{W_s}{\sqrt{s}}, \frac{W_t}{\sqrt{t}}) = \frac{\min(s, t)}{\sqrt{st}},$$

which is not a function of $|t-s|$. For example, the covariance for $s=1, t=2$ is not the same as the covariance for $s=2, t=3$. Therefore this is neither.

c. $e^{-\frac{t}{2}} W_{e^t}, t \geq 0$ (Note: this is Brownian motion evaluated at time e^t for each t).

This is the Ornstein-Uhlenbeck process. It is strongly stationary.

d. $W_t - tW_1, 0 \leq t \leq 1$.

This is the Brownian bridge. It is neither since its covariance function is $\sigma(s,t) = s(1-t)$ which is not a function of $|t-s|$.