Math 478
Spring 2019
Midterm exam 2
4/21/19

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	0	
Total:	100	

1. After midterm 1, the number of students attending a particular Math 478 lecture has been distributed as a Poisson(10) random variable. The distribution of the majors has also changed. Now a randomly selected student from Math 478 has 1/2 chance of being a Math major, 1/3 chance of being a Stat major, 1/12 chance of being an Engineering and 1/12 change of being a Business major.

   (a) (10 points) Find the probability that 5 Math majors, 4 Stat majors, 3 Engineering majors and 2 Business majors attend a particular lecture.

   Ans: This is the Yoga studio problem, generalized to n categories. In the abstract, let X_1 be the number of Math majors, X_2 Stat, X_3 Engineering and X_4 Business. Let the corresponding probabilities be $p_i, i = 1 \cdots 4$. Let $\lambda = 10$ as the mean of the total attendance $\sum_i X_i$. Then we're asking for

$$\begin{aligned}
& P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4, \sum_i X_i = \sum_i x_i) \\
= \; & P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4 \Big| \sum_i X_i = \sum_i x_i) P(\sum_i X_i = \sum_i x_i) \\
= \; & \frac{(\sum_i x_i)!}{\prod_i x_i!} \prod_i p_i^{x_i} e^{-\lambda} \frac{\lambda^{\sum_i x_i}}{(\sum_i x_i)!} \\
= \; & \prod_i e^{-\lambda p_i} \frac{(\lambda p_i)^{x_i}}{x_i!},
\end{aligned}$$

   with of course $x_1 = 5, x_2 = 4, x_3 = 3, x_4 = 2$. This also shows that the numbers of attendance of students from each major are independent Poissons with mean $(\lambda p_i), i = 1, \cdots, 4$.

   (b) (10 points) Find the variance of the number of Math majors attending a particular lecture.

   Because the variance of a Poisson (λp_1) is λp_1, the variance is 10 (0.5) = 5.

2. (20 points) Professor T.'s lecture can be classified as either head-scratching or mind-blowing. If his previous two lectures were head-scratching then the next lecture will be head-scratching with probability 0.6; if his previous two lectures were one head-scratching and one mind-blowing (in either order) then the next lecture will be head-scratching with probability 0.3; finally if his previous two lectures were mind blowing then the next lecture will be mind blowing with probability 0.7. Suppose that professor T's lectures last week's Tuesday and Thursday were mind-blowing. What's the probability that his lecture will be head scratching the Thursday of this week?

   Ans: This is the rain - shine problem. If we denote H for head-scratching and M for mind blowing and the followings

$$\begin{aligned}
1 &: HH \\
2 &: MH \\
3 &: HM \\
4 &: MM
\end{aligned}$$

   then the lecture quality of 2 consecutive classes can be captured as a Markov chain with state space $\{1,2,3,4\}$ and transition probability

$$P = \begin{bmatrix} .6 & 0 & .4 & 0 \\ .3 & 0 & .7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}.$$

The answer is then $P^2_{41} + P^2_{42}$. It can also be calculated directly to give .3 as the final answer.

3. Professor X. (from the Academy for Gifted Youngsters) has the exceptional ability to change real world probabilities. Specifically, he alternates a die's probabilities so that its next toss result depends on its previous toss result. If we let the die's kth toss be represented by Y_k then we have $P(Y_{k+1} = j | Y_k = i) = P_{ij}, i, j = 1, \cdots, 6$ where P is given as followed

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0 & 0.2 & 0.3 & 0 & 0.1 \\ 0.5 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0 & 0.3 & 0 & 0.4 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

In the following, give your answers in terms of the power of some matrix Q that you specify.

(a) (10 points) Professor X.'s first toss is 2. What is the probability that at least 1 of his next 10 tosses is an odd number?

Ans : This is an absorbing state problem with all odd states being treated as one absorbing state (class). The corresponding transition matrix for the new chain is

$$Q = \begin{bmatrix} 0 & .3 & .1 & .6 \\ .1 & .1 & .2 & .6 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here the corresponding classes represented by Q are $2, 4, 6, odd$ in that order, where odd is our absorbing state. The answer is then $Q^{10}_{2\ \text{odd}}$.

(b) (10 points) Professor X.'s first toss is 3. What is the probability that none of his next 20 tosses is odd?

Here we also treat all odd states as one absorbing class and reuse the matrix Q in part a. On the other hand, we first calculate the complementary quantity p, which is the probability that some of his next 20 tosses is odd. Since 3 is in the absorbing class, we need to do a first step a analysis. Thus we have

$$p = \sum_{i\ \text{odd}} P_{3i} + \sum_{i\ \text{even}} P_{3i} Q^{19}_{i\ \text{odd}}.$$

The answer is then 1 - p.

4. It turns out that professor T.'s training was not good enough for the underwater yoga class at the Rutgers Recreation Center. Thus he decides to change his training routine as followed. On each day, he tosses a fair die and trains holding his breath for X minutes where X is a Uniform [0, k] RV and k is the result of the die toss (from 1 to 6).

   (a) (10 points) On a particular day, what is the probability that professor T's training session is less than 2 minutes?

   Ans: Let Y be the result of the die toss. Then

$$\begin{aligned} P(X < 2) = E(P(X < 2|Y)) &= \sum_{i=1}^{6} \frac{1}{6} P(X < 2|Y = i) \\ &= \frac{1}{6}(1 + \sum_{i=2}^{6} \frac{2}{i}). \end{aligned}$$

   (b) (10 points) What are the expectation and the variance of professor T's daily training time?

$$E(X) = E(E(X|Y)) = E(\frac{Y}{2}) = \frac{7}{4}.$$

   We also have

$$\begin{aligned} Var(E(X|Y)) = Var(\frac{Y}{2}) &= \frac{1}{4}(E(Y^2) - E^2(Y)) \\ E(Var(X|Y)) = E(\frac{Y^2}{12}) &= \frac{1}{12}E(Y^2). \end{aligned}$$

   Thus

$$\begin{aligned} Var(X) &= E(Var(X|Y)) + Var(E(X|Y)) = \frac{1}{3}E(Y^2) - \frac{1}{4}E^2(Y) \\ &= \frac{1}{18}\sum_{i=1}^{6} i^2 - \frac{1}{24}(\sum_{i=1}^{6} i)^2. \end{aligned}$$

5. (20 points) Professor T. receives papers to review from various journals. Suppose that X, the number of papers he receives per week for review is a Poisson(5) RV. For each paper he receives, professor T takes Y weeks to finish, where Y is a Geometric(1/3) RV. Assume independence in all RVs involved in this problem. On the first week of January 2019, professor T. received 3 papers. What is the (approximate) probability that during the last week of December, 2019 professor T has less than 2 papers for review to finish?

   Ans: This is the hotel problem. After 56 steps, the Markov chain (the number of papers being reviewed at the kth week) should be reaching the limiting distribution, which is also the stationary distribution. We showed that this limiting distribution is Poisson(λ/p) = Poisson(15). Therefore the probability is

$$e^{-15} + 15e^{-15} = 16e^{-15}.$$

6. (Extra credit - 10 points) Using the same context as question 3 in this exam, and suppose that the die is tossed for infinitely many times. Find the average length of time that professor X. gets consecutive even tosses.

   Ans: This is the rate of breakdown problem part b where we view the even states as the "up" states. Thus the answer is

$$\frac{\sum_{i \text{ even}} \pi_i}{\sum_{i \text{ even}, j \text{ odd}} \pi_i P_{ij}},$$

   where we should check that the chain is ergodic and π is the left eigenvector of P with eigenvalue 1 and sum to 1.