Math 478
Fall 2018
Midterm exam 1
2/21/19

Name (Print): ______________________

This exam contains 4 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
7	0	
Total:	100	

1. Students from Math 478 equally likely come from 4 different majors : Math, Statistics, Engineering and Business. In the Fall 2018 semester, there were 10 students enrolled in Math 478.

   (a) (10 points) Find the expected number of different majors in Math 478 of Fall 2018.

   Ans: This is one of the coupon problems, Ross 2.56.

   Let X_i be 1 if the ith major is among the 10 students, $i = 1, \cdots, 4$ and 0 otherwise. Then the number of different majors in the class is $X = \sum_{i=1}^{4} X_i$. Now

$$\begin{aligned} E(X_i) &= P(X_i = 1) = 1 - P(X_i = 0) \\ &= 1 - P(\text{ no student in the class is of the ith major }) \\ &= 1 - (1 - 1/4)^{10} = 1 - (3/4)^{10}. \end{aligned}$$

   Therefore $E(X) = 4(1 - (3/4)^{10})$.

   (b) (5 points) Is it guaranteed that all 4 majors will be present in Math 478, Fall 2018? Explain.

   Ans: No, either by recognizing that $E(X) < 4$ or by the fact that $P(X_i = 0) = (3/4)^{10} > 0$.

2. (15 points) In the Spring 2019 semester, there's a sudden surge of interest in Math 478. There are (theoretically) infinitely many students waiting to enroll in the class. So the undergraduate office takes a different approach : they would enroll students randomly on a one on one basis until 4 different majors are present in the course. Again, assuming that students equally likely come from 4 different majors as the previous problem. Find the expected number of students in Math 478 of Spring 2019.

   Ans: This is also one of the coupon problems, Ross 2.42.

   Let X_i be the number of students we admit before the ith major appears. Then $X_1 = 1, X_2$ has a Geometric (3/4) distribution since we are waiting for a student NOT of the first major, X_3 has a Geometric (2/4) and X_4 has a geometric (1/4) distribution. Thus the total number of students is $X = \sum_{i=1}^{4} X_i$ with expectation $1 + 4/3 + 2 + 4 = \frac{25}{3}$, which is less than 10 !

3. (20 points) The instructor of Math 478, professor T. reads 2 math books a day for entertainment. His first book, from Typo-free publication, has typos that are distributed as a Poisson random variable at a rate of 0.05 typo per page. His second book, from First-draft accepted printing house, has typos that are distributed as a Poisson random variable at a rate of 2 typos per page. Yesterday, professor T. read 10 pages from the first book and 2 pages from the second one and found a total of 7 typos ! What is the probability that all 7 come from the second book? (Typos on different pages and different publishers are independent.)

   Ans: If we let X_i be the typos on one page of the first book then X_i are iid Poisson (0.05). Similarly, let Y_i be the typos on one page of the second book then Y_i are iid Poisson (2). The total typos on the first book from yesterday is $X = \sum_{i=1}^{10} X_i$ which is a Poisson(0.5) distribution. Similarly, the total typos on the second book from yesterday is $Y = \sum_{i=1}^{2} Y_i$ which is a Poisson(4) distribution. Now the distribution of $Y | X + Y = 7$ is a Binomial(7, 4/4.5). The probability that it is equal to 7 is $(4/4.5)^7$.

4. Professor T. has a biased coin that has probability p of flipping H. Using this coin, he plays a game with professor X. (from the Academy for Gifted Youngsters). They each take turns to flip the coin until the results of two consecutive tosses are different.

(a) (10 points) Find the expected number of tosses in the game (Hint: condition on the first two flips. You will not receive full credit for this question if your answer is in the form of an infinite series).

Let N be the number of tosses and X_{12} be the result of the first two tosses. Then

$$\begin{aligned} E(N) &= E(N|X_{12}=HH)p^2 + [E(N|X_{12}=HT) + E(N|X_{12}=TH)]p(1-p) \\ &+ E(N|X_{12}=TT)(1-p)^2. \end{aligned}$$

Now $N|X_{12}=HH$ has $2+N_1$ where N_1 is a Geometric $(1-p)$ distribution (we're just waiting for a tail to appear). Similarly $N|X_{12}=TT$ has a $2+N_2$ distribution where N_2 is a Geometric (p). Therefore

$$E(N) = p^2(2+\frac{1}{1-p}) + 4p(1-p) + (1-p)^2(2+\frac{1}{p}).$$

(b) (5 points) Professor T. can bet on the outcome of the game as whether the last toss will be H or T. Suppose that $p = 1/3$. Which outcome should professor T. bet on to maximize his chance of winning and why?

Ans: This is Ross 2.28 part b.

If the first toss is H then it's guaranteed the last toss is T. Similarly, if the first toss is T then it's guaranteed the last toss is H. Since $P(T) = 2/3 > P(H) = 1/3$, he should bet on the last toss being H.

5. (15 points) Using the same biased coin as in the previous problem, professor T. organizes a coin tossing contest between 6 students in Math 478 and 8 students in Math 244. Each student takes turn flipping the coin and the team that tosses more H wins. There were 3 Hs (and thus 11 Ts) in total. What is the probability that team 478 wins this contest?

Ans: Let X_1 be the number of H's in team 478 and X_2 be the number of H's in team 244. Then $X_1|X_1+X_2=3$ is a Hypergeometric(6,8,3). The probability that team 478 wins is the probability that this RV takes on values 2 and 3, which is

$$\frac{\binom{6}{2}\binom{8}{1}}{\binom{14}{3}} + \frac{\binom{6}{3}}{\binom{14}{3}}.$$

6. After finish grading the first midterms, professor T. returns them in class. In a hurry to start new exciting materials, professor T. just distributes the exam randomly among the students. There are 45 students in Math 478.

(a) (10 points) Find the expected number of students that get their correct exam back.

This is the hat problem. Let $X_i = 1$ if the ith student gets the correct exam and 0 otherwise. Then the total number of students getting their correct exams is $X = \sum_{i=1}^{45} X_i$. Since $E(X_i) = 1/45$ we have $E(X) = 1$ by linearity.

(b) (10 points) Find the variance of the number of students that get their correct exam back.

We have

$$Var(X) = \sum_{i=1}^{45} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j).$$

Now $Var(X_i) = \frac{1}{45}\frac{44}{45} = \frac{44}{45^2}$.

$$E(X_iX_j) = P(X_i = 1, X_j = 1) = \frac{43!}{45!} = \frac{1}{44 \times 45}.$$

Thus

$$\begin{aligned} Cov(X_i, X_j) &= E(X_iX_j) - E(X_i)E(X_j) = \frac{1}{44 \times 45} - \frac{1}{45^2} \\ &= \frac{1}{44 \times 45^2}. \end{aligned}$$

Thus

$$\begin{aligned} Var(X) &= \sum_{i=1}^{45} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \\ &= 45\frac{44}{45^2} + 2\binom{45}{2}\frac{1}{44 \times 45^2} \\ &= \frac{44}{45} + \frac{1}{45} = 1. \end{aligned}$$

7. (Extra credit) Professor T. plans to enroll in underwater yoga at the Rutgers Recreation Center in March. To prepare for the class, professor T. trains to hold his breath each day. Suppose that the time he holds his breath is distributed as U[2,4] (minutes) and he trains once a day for 7 days.

   (a) (5 pts) Each time professor T. holds his breath longer than all the previous sessions he awards himself a cookie. Find the expected number of cookies professor T. has by the end of training.

   Ans: This is the record problem, Ross 2.72 a and b, except we don't count the first record. Let $R_i = 1$ if a record happens at time $i, i \geq 2$. Then the total number of cookies is $R = \sum_{i=2}^{7}$ with $E(R_i) = 1/i$. Therefore the answer is

$$\sum_{i=2}^{7} \frac{1}{i}.$$

   (b) (5 pts) Find the variance of the number of cookies professor T. has by the end of training.

   Ans: From the homework problem, it is

$$\sum_{i=2}^{7} \frac{i-1}{i^2},$$

   since subtracting 1 away from the expression doesn't change the variance.