# Markov chain in discrete time II

Math 478

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## 1 Classification of states

A state $j$ is **accessible** from state $i$ if $P_{ij}^n > 0$ for some $n \geq 0$. Conversely, if $j$ is NOT accessible from $i$ (that is $P_{ij}^n = 0, \forall n$) then the chain will never enter $j$ starting from $i$:

$$\begin{aligned} P(X_k = j, \text{ some } k \geq 1 | X_0 = i) &= P(\cup_{k=1}^{\infty}\{X_k = i\}|X_0 = i) \\ \leq \sum_{i=1}^{\infty} P(X_k = j|X_0 = i) = 0. \end{aligned}$$

Finally, if state $j$ is accessible from state $i$ it does NOT guarantee that the chain will enter state $j$ at some point. That is we can have

$$P(X_k = j, \text{ some } k \geq 1|X_0 = i) < 1$$

even if state $j$ is accessible from state $i$. Exercise : give an example of such a case.

Two states $i, j$ are said to **communicate** if $i$ is accessible from $j$ and vice versa. In this case, we denote it as $i \leftrightarrow j$. It is clear that $\leftrightarrow$ forms an equivalence relation, that is:

a) $i \leftrightarrow i$ (this comes from the fact that we allow $n$ to be 0 in the accessible definition).

b) $i \leftrightarrow j$ implies $j \leftrightarrow i$.

c) $i \leftrightarrow j, j \leftrightarrow k$ implies $i \leftrightarrow k$.

The state space is partitioned into equivalence classes under this equivalence relation. A Markov chain is **irreducible** if it only has one such class.

A state $i$ is **recurrent** if it is guaranteed that the process will return to state $i$ starting from $i$

$$f_i \stackrel{\triangle}{=} P(X_k = i, \text{ some } k \geq 1|X_0 = i) = 1.$$

A state $i$ is **transient** otherwise. Given a state $i$, let $N_i$ be the number of visits to the state starting at time 0, given that $X_0 = i$. That is

$$N_i = \sum_{k=0}^{\infty} I_k,$$

where $I_k = 1$ if $X_k = i | X_0 = i$ and 0 otherwise. Starting at state $i$, the process either never re-enters $i$ with probability $1 - f_i$ or it will re-enter with probability $f_i$. If it re-enters state $i$ we can repeat the process again. So if we look at NOT returning to state $i$ as a "success" then this can be viewed as a Geometric distribution with "success" probability $1 - f_i$. The expected number of visits in this case is

$$E(N_i) = \frac{1}{1 - f_i} < \infty$$

Note that $f_i < 1$ for this case to make sense. This also implies that the chain will visit $i$ only finitely many times *if and only if* it is transient. On the other hand, letting $f_i \uparrow 1$ shows that $E(N_i) = \infty$ if the state is recurrent. That is the chain will re-visit state $i$ infinitely many times *if and only if* it is recurrent. An important consequence here is that in a finite state space Markov chain, not all states can be transient (since the chain has to visit some state(s) infinitely many times in the infinite time horizon).

We also have

$$\begin{aligned} E(N_i) &= \sum_{k=0}^{\infty} E(I_k) = \sum_{k=0}^{\infty} E(I_k) \\ &= \sum_{k=0}^{\infty} P_{ii}^k. \end{aligned}$$

We thus have the following proposition.

**Proposition 1.1.** *A state $i$ is*

$$\begin{aligned} \textit{recurrent if} \sum_{k=1}^{\infty} P_{ii}^k &= \infty \\ \textit{transient if} \sum_{k=1}^{\infty} P_{ii}^k &< \infty. \end{aligned}$$

Note that the sum in the above starts at $k = 1$ instead of 0 but it doesn't change anything because $P_{ii}^0 = 1$.

Now let state $i$ be recurrent and suppose it also communicates with state $j$. This means $P_{ij}^m > 0$ and $P_{ji}^n > 0$ for some $m, n$. Then

$$\sum_k P_{jj}^k > \sum_k P_{ji}^n P_{ii}^k P_{ij}^m = \infty.$$

Thus $j$ is also recurrent. Note that intuitively this is not obvious because we stated that if $i \leftrightarrow j$ it is NOT guaranteed that starting from $i$ the chain will ever visit $j$ and vice versa (again construct an example here.) The fact that $i$ is recurrent changes this result. On the one hand, since starting from $i$ we have a positive chance of visit $j$ after some time step, and we are guaranteed to return to $i$, we are guaranteed to visit $j$ starting from $i$ (because if we "failed" to visit $j$ starting from $i$ at some turn we can just revisit $i$ and try again, like a Geometric distribution). That is

$$P(X_k = j, \text{ some } k \geq 1 | X_0 = i) = 1.$$

Now starting from $j$, why must we reach $i$ ? Suppose this is not the case, then first staring from $i$ we are guaranteed to visit $j$. But if we are not guaranteed to visit $i$ from $j$ then we cannot guaranteed a return to $i$ starting from $i$. That is $i$ is transient, a contradiction.

The above result also implies that recurrence and transience is a **class property.** That is given a Markov chain, we can partition the state space into recurrent and transient classes. Finally, an irreducible Markov chain with finite state must be recurrent.

**Example 1.2.** *Transience / recurrence property of 1 dimensional random walk - Analytical proof*

Let $X_i$ be iid with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$. Let $S_k = \sum_{i=1}^k X_i$. Then $S_k$ is a Markov chain with state space being the integers $\mathbb{Z}$. It is easy to check that $i \leftrightarrow j$ for any $i, j \in \mathbb{Z}$ and so $S_k$ is irreducible. It remains to determine if $S_k$ is transient or recurrent. For that we only need to focus on one state $i = 0$. We need to compute

$$\sum_{k=1}^{\infty} P_{00}^k = \sum_{k=1}^{\infty} P_{00}^{2k},$$

since we cannot return to 0 after an odd number of steps. Now given a $k$ we have

$$P_{00}^{2k} = \binom{2k}{k} p^k (1-p)^k,$$

since we need to choose the times for the k steps to the right (and the remaining for the left step). Thus

$$\sum_{k=1}^{\infty} P_{00}^{k} = \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k.$$

The convergence of this series clearly only depends on the behavior of the terms when $k$ is large. In this case we can use the Stirling's approximation for $k$ large:

$$k! \approx k^{k+\frac{1}{2}} e^{-k} \sqrt{2\pi}.$$

Then

$$\begin{aligned} \binom{2k}{k} &\approx c\frac{(2k)^{2k+\frac{1}{2}}e^{-2k}}{k^{2k+1}e^{-2k}} \\ &= c2^{2k+\frac{1}{2}}k^{-\frac{1}{2}} \\ &= c4^k k^{-\frac{1}{2}} \end{aligned}$$

and

$$\binom{2k}{k} p^k (1-p)^k \approx c\,(4p(1-p))^k\, k^{-\frac{1}{2}}$$

Thus the series is approximately a geometric series and it converges if and only if the ratio is less than 1 :

$$4p(1-p) < 1.$$

This happens if and only if $p \neq 1/2$ since $4p(1-p) \leq [p + (1-p)]^2 = 1$ with equality when $p = 1 - p$. Our conclusion is a one dimensional symmetric random walk ($p = 1/2$) is recurrent and is transient otherwise. One can also show, using a similar proof as above, that a two dimensional symmetric random walk is recurrent. Remarkably, symmetric random walk of dimension $\geq 3$ are all transient.

## 2 Limiting distribution

Let $i$ be a recurrent state of a Markov chain $X_k$. Then we know

$$\sum_{n} P_{ii}^{n} = \infty.$$

Recall that if for a non-negative series $a_n$, $\sum_n a_n < \infty$ implies $\lim_{n\to\infty} a_n = 0$. We want to ask a similar question here for $\lim_{n\to\infty} P_{ii}^n$ ( It turns out that this question is a particular case of the more general question about $\lim_{n\to\infty} P_{ij}^n$ for a recurrent state $j$, see below ). Note that the fact $\sum_n P_{ii}^n = \infty$ alone does not give any information about this limit, even whether or not it exists. On the other hand, if state $i$ is recurrent then the chain visits it infinitely many times. So it seems like $\lim_{n\to\infty} P_{ii}^n$ represents the proportion of times the chain visits state $i$ compared with other recurrent states. This is the right intuition, but we need to make it more precise with some further concepts.

For any given state $i$, define

$$\tau_i := \min\{k > 0, X_k = i | X_0 = i\},$$

to be the first return time to state $i$, starting from $i$. A (recurrent) state $i$ is *positive recurrent* if the expectation of its return time $E(\tau_i)$ is finite and is *null recurrent* if $E(\tau_i)$ is infinite.

**Example 2.1.** *Recurrence type of one dimensional random walk*

We have discussed that in a symmetric 1-d random walk, 0 is a recurrent state. To check whether 0 is null or positive recurrent, we may look at $E(\tau_0)$, where $\tau_0$ is the first time the chain returns to 0. Now $P(\tau_0 = 2k)$ is NOT $P_{00}^{2k} = \binom{2k}{k}\frac{1}{2^{2k}}$ as the example above. The reason is $P_{00}^{2k}$ includes many instances of the chain revisiting 0 BEFORE the time $2k$ (see also the remark below). We may use the law of inclusion / exclusion to calculate $P(\tau_0 = 2k)$ (by subtracting all the possbilities of the chain returning to 0 before time $2k$ from $\binom{2k}{k}$). On the other hand, this leads to complicated calculation that reveals little intuition. We present an easier way to do so with the limiting probability below (which also demonstrates a practical use of the concept).

Note that for a Markov chain, $P_{ii}^n$ is not necessarily the **first** time the chain revisits state $i$ after n steps. Otherwise, from the fact that $\sum_n P_{ii}^n = \infty$ we must conclude that $\sum_n nP_{ii}^n = \infty$ and thus every recurrent state is null recurrent, which is evidently not true with an easy counter example. In fact, *in a finite state Markov chain, every recurrent state is positive recurrent.* Moreover, positive recurrence is a *class property.*

For a given state $i$, we say it is periodic with period $d$ if $P_{ii}^n = 0$ for $n \bmod d \neq 0$. That is state $i$ has period $d$ if starting from $i$ we can return to $i$ only after a multiple of $d$ steps. In the 1-d random walk, it is clear that all states have period 2. If $d = 0$ we say $i$ is aperiodic. Periodicity is also a *class property.*

Finally, a class of states is *ergodic* if it is positive recurrent and aperiodic. We have the following important theorem:

**Theorem 2.2.** *Convergence of ergodic chain transition probabilities*

*Let $X_k$ be an irreducible ergodic Markov chain (with possibily countably many states). Then for any $i, j$*

$$\lim_{n\to\infty} P_{ij}^n \textit{ exists and equals } \pi_j.$$

*Moreover, $\pi_j$ is the unique solution to the system*

$$\begin{aligned} \pi_j &= \sum_i \pi_i P_{ij} \\ \sum_j \pi_j &= 1. \end{aligned}$$

*That is $\pi$ is the left eigenvector of the transition matrix $P$ with eigenvalue 1. $\pi$ is also referred to as the* **stationary** *probability of the Markov chain $X_k$. The reason is if we start out with initial distribution $\pi$, that is $P(X_0 = i) = \pi$, all subsequent distribution will also be $\pi : P(X_k = i) = \pi_i, \forall k$.*

We have the following remarks.

- The most important result in the above theorem is the convergence of $P_{ij}^n$. The other facts follow easily from this convergence. In fact, from the Kolgomorov equation

  $$P_{ij}^{n+1} = \sum_k P_{ik}^n P_{kj},$$

  taking limit as $n \to \infty$ on both sides we obtain $\pi_j = \sum_i \pi_i P_{ij}$.

- If we write the Kolmogorov equation the other way

  $$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n,$$

  then we actually get $\pi_j = \sum_k P_{ik} \pi_j$, which is just a reflection of the fact that the row sum of a transition probability matrix is 1. Thus it is emphasized that the limiting probability is a left and not a right eigenvector of the transition matrix.

- From the fact that $\sum_j P_{ij}^n = 1$ and $\lim_{n\to\infty} P_{ij}^n = \pi_j$ independent of $i$ we derive $\sum_j \pi_j = 1$. (Here it actually requires an interchange of summation and limit, which always holds when the sum if finite in the case of finite Markov chain. This connects to the fact that a finite state ergodic Markov chain is always positive recurrent).

- The above discussion also explains why $\lim_{n\to\infty} P_{ij}^n$ only depends on $j$ and not on $i$. First we can see it from an intuitive point of view. We explained the meaning of the limit as the proportion of the time the chain spends in $j$ in the long run and thus it should be only a function of $j$. Second, this reflects the Markov property of the chain. Markov property can be informally described as "finite memory" property of the chain. If the limiting probability still depends on $i$ then the chain has "infinite memory" which is a contradiction to the finite memory property.

- In terms of mathematics, we first can dismiss the idea that the limit $\lim_{n\to\infty} P_{ij}^n$ is a function of both $i, j$. If that is the case then from the Kolmogorov equation we'll have $\bar{P} = \bar{P}P = P\bar{P}$ where $\bar{P}$ is the limit of the transition probability matrix $P$. This can only be true $P$ is an identity matrix. On the other hand, if the limit is a function of $i$ then there are two consequences. First, the limiting distribution $\pi$ is a right eigenvector of $P$ (which again explains why in the true scenario we must emphasize $\pi$ as the left eigenvector of $P$) from taking the limit of the Kolmogorov equation

$$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n.$$

Second, taking the limit in the "other" Kolmogorov equation

$$P_{ij}^{n+1} = \sum_k P_{ik}^n P_{kj},$$

gives $\pi_i = \sum_k \pi_i P_{kj}$, which cannot be true since the column sum of $P$ is not 1.

- The requirement that the state is aperiodic is intuitively clear. If state $i$ has period $d$ then we can only discuss $P_{ii}^n$ where $n$ is a multiple of $d$. Thus the limit $\lim_{n\to\infty} P_{ii}^n$ must be taken along the subsequence of $n$ that is a multiple of $d$. The general limit $\lim_{n\to\infty} P_{ij}^n$ for any $i, j$ may not exist in the rigorous sense of limit definition (e.g. it converges to one limit when $n$ is even and another when $n$ is odd).

- The requirement that the state is positive recurrent is also clear. If a state $i$ is null recurrent then even though the chain is guaranteed to revisit $i$ and it will visit $i$ infinitely often, the average time it takes to do so for the first time is infinite ! Thus the long run proportion of time the chain spends on state $i$ is 0 or $\pi_i = 0$.

- The statement that an irreducible recurrent finite state Markov chain is positive recurrent can be proven by showing that the transition probability matrix $P$ has a positive left eigenvector with eigenvalue 1. This can be viewed as a linear algebra exercise (albeit a nontrivial one). The irreducible property of the chain can be captured by stating that $P$ cannot be permuted into a block-diagonal matrix.

- The previous statement can be generalized to any Markov chain. That is a Markov chain is irreducible positive recurrent if and only if it has a positive left eigenvector with eigenvalue 1. The irreducibility is captured by the fact that all entries of the eigenvector is non-zero. As a particular example, one can show that the 1-d random walk is null recurrent by showing that the left eigenvector with eigenvalue 1 is the zero vector.

**Example 2.3.** *Null recurrence type of one dimensional random walk*

We now can demonstrate the null recurrence property of one dimensional random walk by showing that the left eigenvector $\pi$ of its transition matrix is the 0 vector. From the symmetry of the situation, we conclude that if there is such a vector $\pi$, it must happen that $\pi_i = \pi_j, \forall i \neq j$. A formal argument may be as followed : consider another Markov chain $Y_k$ where the state space of $Y$ is the state space of $X$ plus a given integer, say $N$. $Y$ and $X$ has exactly the same transition probability matrix and thus must have the same stationary distribution. But this means $\pi^X_{i+N} = \pi^X_i, \forall N$. It then follows easily that $\pi^X_i = 0, \forall i$.

**Example 2.4.** *Stationary distribution of the hotel family problem*

In the hotel problem, it is not a good idea to find the stationary distribution of the number of families staying in the hotel in any one day by solving the equation $\pi P = \pi$, if only for the reason that the transition probabilities obtained for $P$ are not simple expressions. On the other hand, we know how the number of families changes from one day to the next. Using the idea that the stationary distribution is unique, we can just try to guess the distribution that would make our process stationary. It is not unreasonable to guess that the stationary distribution is a Poisson ($\theta$) for some value of $\theta$. This comes from the fact that the number of families arrive each day is a Poisson and the sum of independent Poissons is Poisson. If our guess is correct, we just need to find $\theta$ so that if $X_0 \sim$ Poisson ($\theta$) then $X_1 \sim$ Poission($\theta$).

Again the idea is to avoid using the transition probability so we don't want to compute $P(X_1 = j|X_0 = i)$. On the other hand, we know the number of families arriving the next day $Y_1$ is Poisson ($\lambda$) independent of $X_0$. Thus we know $X_0 + Y_1$ has a Poisson ($\lambda + \theta$) distribution. If the number of families leaving $Z_1$ is also a Poisson ($\lambda$) distribution then we're done (since $Z_1$ is independent of $X_0, Y_1$. We know that $Z_1|X_0$ has a Binomial($X_0, p$) distribution. Thus the unconditional distribution of $Z_1$ is a Poisson $p\theta$ (from the discussion of the Yoga studio problem, Ross example 3.23). Thus we need $p\theta = \lambda$ or the stationary distribution is Poisson ($\frac{\lambda}{p}$).

# 3 Average time spent in a certain state

The stationary distribution $\pi$ represents the percentage time the chain spends at state $i$ in the long run. In this section we make this idea precise. First we have the following proposition.

**Proposition 3.1.** *Let $X_k$ be a irreducible positive recurrent Markov chain and $\pi$ its stationar distribution. Then with probability 1*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(X_i)}{n} = \sum_i \pi_i f(i),$$

*for any bounded function $f$.*

The above result is referred to as an "ergodic" type of result. It basically states that under a certain condition, (long run) time average can be replaced by spatial average. The law of large number can be viewed as one particular case of this statement (even though the sense of convergence may be different, and stronger in the law of large number). In fact, let $X_i$ be an iid sequence of discrete RVs with distribution represented by a RV $X$. Then $X_i$ is a Markov chain with transition probability $P_{ij} = P(X = j)$ ! The stationary distribution $\pi_i$ is the distribution of $X$ itself. And thus applying the above proposition with $f(i) = i$ we see that the long run average converges to $E(X)$.

The proof of the proposition goes as followed. Let $a_i(n)$ be the time $X_k$ spends at state $i$ for the time duration $0, 1, \cdots, n$. Then

$$\sum_{i=1}^{n} f(X_i) = \sum_{i=1}^{\infty} f(i) a_i(n).$$

Note that the LHS is a sum over time while the RHS is a sum over state space. Dividing both sides by $n$ and note that $\lim_{n\to\infty} \frac{a_i(n)}{n} = \pi_i$ we have the result.

**Example 3.2.** *Transition rate, average time spent in a certain state*

Let $X_k$ be an ergodic finite state Markov chain with two types of states : acceptable $A$ and unacceptable $A^c$.$X_k$ can be thought of as a production process where time spent in $A$ represents the "up" time and the time spent in $A^c$ represents the down time.

a. Find the rate at which the production process goes from up to down ( that is, the rate of break down)

b.The average length of time the process remains down when it goes down

c. The average length of time the process remains up when it goes up

Ans: The main challenge of this problem is to translate the language of the Markov chain to the language used in the problem. First we know that there exists a stationary probability, which represents the *proportion* of time the process spends in a certain state in the long run. For that matter, we also know the *proportion* of time the process spends in the acceptable class $A$ and the unacceptable class $A^c$ in the long run. But at least for now we do not have information on the *length of time* the process spends in these states.

a. First a breakdown is a transition from good to bad state. Second, the unit of breakdown *rate* should be the number of breakdown per unit of time. A unit of time in the discrete context is just one step of the Markov chain. In this sense, it seems that the answer is just $\sum_{i\in A, j\in A^c} P_{ij}$. To give an easy example, suppose the chain only has two states, 0 for unacceptable and 1 for acceptable. Then it seems that the answer is $P_{10}$ because in a unit of time, there is $P_{10}$ "breakdowns" happening.

On second thought this answer does not make sense. $P_{ij}$ is a conditional probability. The above sum reads as $\sum_{i\in A, j\in A^c} P_{ij} = \sum_{i\in A, j\in A^c} P(X_{k+1} = j | X_k = i)$. This sum is not a quantity we can make sense of because we sum over all possible starting states. Going back to our simple example, we see that $P_{10}$ only gives the probability of breakdown IF we are in the acceptable state. It does not give us the probability of breakdown in general (because in general we may also be in an unacceptable state and in that state "break down" does not apply). In other words, we need the unconditional probability of being in an unacceptable state starting from an acceptable state. Thus the answer is $\sum_{i\in A, j\in A^c} \pi_i P_{ij}$.

To see why this answer makes intuitive sense, we see that $\pi_i$ represents the proportion of time the chain spends in the acceptable state. Another way to see it is in the long run, the chain will settle in the stationary distribution of $\pi$. From there, the breakdown rate is just

$$\begin{aligned} & P(\text{ move from an acceptable state to an unacceptable state in 1 unit of time }) \\ = \; & P(X_{k+1} \text{ unacceptable }, X_k \text{ acceptable }) \end{aligned}$$

which is the answer we gave. Finally note that the breakdown rate refers to a stationary quantity (not depending on $k$ or distribution of $X_k$). Thus the use of the stationary distribution for its computation also makes sense.

An alternative approach to solving this problem in a similar spirit to the ergodic theorem presented above is as followed. The average breakdown rate should be

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n f(X_i, X_{i+1})}{n},$$

where we define $f(i,j) = 1$ if $i \in A, j \in A^c$ and 0 otherwise. To use the ergodic theorem, we rewrite

$$\sum_{i=1}^n f(X_i, X_{i+1}) = \sum_{i=1}^n f(Y_i)$$

where $Y_i := (X_i, X_{i+1})$ is a Markov process (verify that indeed $Y_i$ is Markov). The transition probability of $Y_i$ is

$$P^Y_{(i,j),(j,k)} = P(Y_{i+1} = (j,k)|Y_i = (i,j)) = P_{jk}.$$

Because $X_i$ is ergodic, $Y_i$ is also ergodic and its stationary distribution is

$$\pi^Y_{ij} = \pi_i P_{ij}.$$

Indeed $\sum_{i,j} \pi^Y_{ij} = \sum_{i,j} \pi_i P_{ij} = 1$ and

$$\sum_i \pi^Y_{ij} P_{jk} \;\; = \;\; \sum_i \pi_i P_{ij} P_{jk} = \pi^Y_{jk}.$$

Here we note that the above sum is only over $i$ and not $i,j$ because to end up at a particular state $(j,k)$, $Y$ must start at a $(i,j)$ where the freedom is only on the $i$ and not on the $j$. For further clarification, see e.g. Ross example 4.4.

Now applying the ergodic theorem we see that

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{i=1}^{n} f(X_i, X_{i+1})}{n} &= \lim_{n\to\infty}\frac{\sum_{i=1}^{n} f(Y_i)}{n} = \sum_{i,j} f(i,j)\pi_i^Y \\ &= \sum_{i,j} f(i,j)\pi_i P_{ij} \\ &= \sum_{i\in A, j\in A^c} \pi_i P_{ij}.\end{aligned}$$

b. This question is clearly related to part c. It is also clear that the answer is in terms of some unit of time ( or time step in the case of the Markov chain ). What we have now with our Markov chain is proportion and breakdown rate. So it's intuitive that we should relate the times in question with some rate or proportion. If we call $D$ the answer for part b and $U$ the answer for part c then it is clear there is on average 1 breakdown per time interval of length $D+U$. Thus our breakdown rate found in part a is exactly $\frac{1}{D+U}$ :

$$\sum_{i\in A, j\in A^c} \pi_i P_{ij} = \frac{1}{D+U}.$$

Now we need one more equation relating $D, U$. It's not hard to see that we want to relate the proportion of time the chain spends in up or down states to $U$ and $D$. For example, the proportion of time the process spends in the up states would be

$$\frac{U}{D+U} = \sum_{i\in A} \pi_i.$$

Combining the two equations, we have easily

$$\begin{aligned} U &= \frac{\sum_{i\in A} \pi_i}{\sum_{i\in A, j\in A^c} \pi_i P_{ij}} \\ D &= \frac{\sum_{i\in A^c} \pi_i}{\sum_{i\in A, j\in A^c} \pi_i P_{ij}}.\end{aligned}$$

# 4 Gambler's ruin in general

Consider a Markov chain $X_k$ with the following transition probability:

$$\begin{aligned} P(X_{k+1} = i+1 | X_k = i) &= p \\ P(X_{k+1} = i-1 | X_k = i) &= 1-p, i \in \mathbb{Z}.\end{aligned}$$

Thus $X_k$ is a one dimensional random walk with probability $p$ of going one unit to the right and $1-p$ of going one unit to the left in one time step. Suppose $X_0 = 0$. We want to ask the probability

$$P(X_k = n_1 \text{ before } X_k = -n_2 \text{ for some } k | X_0 = 0) = ?$$

This is the ruin probability of a gambler whose initial fortune is $n_2$ playing a game of flipping a coin with probability of H equalling $p$ against another player whose initial fortune is $n_1$. It is clear that we can cast this problem into another Markov chain with $N$ states, where states $1, N$ are absorbing with transition probabilities

$$\begin{aligned} P(X_{k+1} = i+1 | X_k = i) &= p \\ P(X_{k+1} = i-1 | X_k = i) &= 1-p, 1 < i < N. \end{aligned}$$

The connection of this problem to limiting distribution is as followed. We observe that states $2, \cdots, N-1$ are transient so the chain will eventually settle in either state 1 or $N$. The chain is not ergodic so the previous theory of limiting distribution does not apply. On the other hand, we can believe that if we take the limit $P^n$ as $n \to \infty$ where $P$ is the transition matrix of the chain it will converge to a limit whose columns are zeros except the first and last ones. Those represent the absorbing probabilities to state 1 and $N$ respectively. It is our goal in this section to derive these absorbing probabilities in another way.

Denote $\pi_i$ as the absorbing probability to state $N$ starting from state $i$. Then

$$\begin{aligned} \pi_i &= P(X_k = N \text{ before } X_k = 1 \text{ for some } k | X_0 = i) \\ &= pP(X_k = N \text{ before } X_k = 1 \text{ for some } k | X_1 = i+1) + \\ &\quad (1-p)P(X_k = N \text{ before } X_k = 1 \text{ for some } k | X_1 = i-1) \\ &= p\pi_{i+1} + (1-p)\pi_{i-1} \end{aligned}$$

with $\pi_1 = 0$ and $\pi_N = 1$. Since $\pi_1 = 0$ we have

$$\begin{aligned} \pi_3 &= p\pi_2 \\ \pi_3 &= p\pi_4 + (1-p)\pi_2 = p\pi_4 + \frac{(1-p)}{p}\pi_3 \end{aligned}$$

from which we conclude

$$\pi_4 = \frac{p-q}{p^2}\pi_3 = \frac{p-q}{p}\pi_2.$$

We can try one more step to see the pattern, but we can be more efficient looking at the abstract formula:

$$\begin{aligned}\pi_i &= p\pi_{i+1} + (1-p)\pi_{i-1} \\ (1-p)(\pi_i - \pi_{i-1}) &= p(\pi_{i+1} - \pi_i).\end{aligned}$$

That is

$$\begin{aligned}\pi_{i+1} - \pi_i &= \frac{q}{p}(\pi_i - \pi_{i-1}) = \cdots = \left(\frac{q}{p}\right)^{i-1} (\pi_2 - \pi_1) \\ &= \left(\frac{q}{p}\right)^{i-1} \pi_2.\end{aligned}$$

The LHS obviously suggests a telescoping sum, which is

$$\begin{aligned}\sum_{i=1}^{N-1}(\pi_{i+1} - \pi_i) = \pi_2 \sum_{i=1}^{N-1} \left(\frac{q}{p}\right)^{i-1} &= \frac{1 - \left(\frac{q}{p}\right)^{N-1}}{1 - \frac{q}{p}},\ p \neq q \\ &= N-1,\ p = q.\end{aligned}$$

Since the LHS is 1,

$$\begin{aligned}\pi_2 &= \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^{N-1}},\ p \neq q \\ &= \frac{1}{N-1},\ p = q.\end{aligned}$$

Note how this is consistent with the glamber's ruin problem in the symmetric random walk we have derived before. The same computation can be applied to compute $\pi_i$ :

$$\begin{aligned}\pi_i = \sum_{k=1}^{i-1}(\pi_{k+1} - \pi_k) = \pi_2 \sum_{k=1}^{i-1} \left(\frac{q}{p}\right)^{k-1} &= \frac{1 - \left(\frac{q}{p}\right)^{i-1}}{1 - \left(\frac{q}{p}\right)^{N-1}},\ p \neq q \\ &= \frac{i-1}{N-1},\ p = q.\end{aligned}$$

If $p > 1/2$ then $\frac{q}{p} < 1$ and thus

$$\lim_{N\to\infty} \pi_i = 1 - \left(\frac{q}{p}\right)^{i-1}.$$

This is the probability that a gambler can "not lose" against an infinitely rich opponent. Here not losing means he does get arbitrarily rich. Note that this probability is an increasing function of his initial fortune $i$. Also he must have an advantage in winning each game ($p > 1/2$) before he has a hope to do so.

On the other hand, $p < 1/2$ then $\frac{q}{p} > 1$ and thus

$$\lim_{N \to \infty} \pi_i = 0.$$

In fact, if $p = 1/2, \lim_{N \to \infty} \pi_i = 0$ as well. Thus if in a unfavored or fair game, the finite fortune player is guaranteed to go bankrupt against and infinitely rich player.