Conditional variance

Math 478

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1 Overview

Let X, Y be RVs. Then

$$\begin{aligned} Var(X|Y) &= E\big([X - E(X|Y)]^2\Big|Y\big) \\ &= \sum_x xP_{X|Y}(X = x|Y) \text{ (discrete)} \\ &= \int xf_{X|Y}(x|Y)dx \text{ (continuous) .} \end{aligned}$$

By linearity of conditional expectation:

$$\begin{aligned} Var(X|Y) &= E\big([X - E(X|Y)]^2\Big|Y\big) \\ &= E\big(X^2 - 2XE(X|Y) + E^2(X|Y)\Big|Y\big) \\ &= E\big(X^2|Y\big) - 2E(X|Y)E(X|Y) + E^2(X|Y) \\ &= E\big(X^2|Y\big) - E^2(X|Y), \end{aligned}$$

similar to the non-conditional variance formula. The interpretation of $Var(X|Y)$ is that

a) It is a RV, specifically also a function of Y

b) It is the best mean square estimate of the spread of X with respect to $E(X|Y)$, its conditional estimate based on Y.

c) If X, Y are independent then $Var(X|Y) = Var(X)$.

d) $Var(f(X)|X) = 0$ for any function f.

e) One can develop the notion of conditional covariance $Cov(X, Y|Z)$ in a similar manner and thus have a formula for $Var(X + Y|Z)$.

f) In general, the relation $Var(X+Y|Z) = Var(X|Z) + Var(Y|Z)$ requires the notion of independence of X, Y conditioned on Z, which is different from independence of X, Y only. See below for an example.

Finally we have

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y)).$$

This can be proven by observing that

$$\begin{aligned} E(Var(X|Y)) + Var(E(X|Y)) &= E(E(X^2|Y) - E^2(X|Y)) + E(E^2(X|Y)) - E^2(E(X|Y)) \\ &= E(X^2) - E^2(X) = Var(X). \end{aligned}$$

2 Examples

2.1 Variance of a Geometric(p)

Let N be a Geometric(p) RV and X_1 be the result of the first trial. We want to calculate $Var(N)$ using the formula

$$Var(N) = E(Var(N|X_1)) + Var(E(N|X_1)).$$

First

$$Var(N|X_1) = E(N^2|X_1) - E^2(N|X_1).$$

We showed that

$$E(N|X_1) = \mathbf{1}_{X_1=1} + E(N+1)\mathbf{1}_{X_1=0}.$$

It follows that

$$E^2(N|X_1) = \mathbf{1}_{X_1=1} + E^2(N+1)\mathbf{1}_{X_1=0}.$$

A similar argument shows that

$$E(N^2|X_1) = \mathbf{1}_{X_1=1} + E[(N+1)^2]\mathbf{1}_{X_1=0}.$$

Thus

$$Var(N|X_1) = Var(N)\mathbf{1}_{X_1=0}.$$

Note that the above equation can be derived more simply from the observation that

$$N\Big|X_1 \stackrel{d}{=} \mathbf{1}_{X_1=1} + (1+N)\mathbf{1}_{X_1=0}.$$

When conditioned on $X_1, \mathbf{1}_{X_1=1}, \mathbf{1}_{X_1=0}$ are deterministic. Thus

$$Var(N|X_1) = Var(N)\mathbf{1}_{X_1=0}.$$

Finally

$$E(Var(N|X_1)) = Var(N)(1-p).$$

On the other hand,

$$\begin{aligned} E(N|X_1) &= \mathbf{1}_{X_1=1} + E(N+1)\mathbf{1}_{X_1=0} \\ &= E(N)\mathbf{1}_{X_1=0} \end{aligned}$$

and thus

$$Var(E(N|X_1)) = E^2(N)p(1-p) = \frac{1-p}{p}.$$

We have

$$\begin{aligned} Var(N) &= E(Var(N|X_1)) + Var(E(N|X_1)) \\ &= Var(N)(1-p) + \frac{1-p}{p}. \end{aligned}$$

Thus $Var(N) = \frac{1-p}{p^2}$.

2.2 Variance of a compound RV

Let X_i be iid and N independent of X_i. Recall that $S = \sum_{i=1}^{N} X_i$ is a compound RV. We compute $Var(S)$ using the formula

$$Var(S) = E(Var(S|N)) + Var(E(S|N)).$$

Now

$$\begin{aligned} Var(S|N) &= Var(\sum_{i=1}^{N} X_i|N) = \sum_{i=1}^{N} Var(X_i|N) \\ &= NVar(X_1). \end{aligned}$$

Remark 2.1. *The above argument is a bit delicate and it's more than just interchanging the variance and summation. In fact the following is not true, even if* X, Y *are independent:*

$$Var(X + Y|Z) = Var(X|Z) + Var(Y|Z).$$

The reason is the relation also depends on the relations between X, Y *and* Z. *For example take* X, Y *to be indendent Poissons then*

$$Var(X + Y|X + Y) = 0 \neq Var(X|X + Y) + Var(Y|X + Y).$$

In the formula for $Var(S|N)$ *we also have the independence of* X_i *and* N *which gives us the result.*

Thus

$$E(Var(S|N)) \;\;=\;\; E(N)Var(X_1).$$

On the other hand,

$$Var(E(S|N)) = Var(NE(X_1)) = E^2(X_1)Var(N).$$

Therefore,

$$Var(S) = E(N)Var(X_1) + E^2(X_1)Var(N).$$