Conditional expectations

Math 478

February 17, 2019

1 Compute the expectation of Geometric(p)

Let N be a Geometric(p) RV. Let $X_1 = 1$ if the first trial is a success and 0 otherwise. Then

$$\begin{aligned} E(N) &= E(E(N|X_1)) = E(N|X_1=0)P(X_1=0) + E(N|X_1=1)P(X_1=1) \\ &= E(N+1)(1-p) + p. \end{aligned}$$

Here we remark that $E(N|X_1 = 1) = E(N + 1)$ can be viewed from the fact that

$$N\Big|(X_1 = 0) \overset{d}{=} N + 1.$$

For example

$$\begin{aligned} P(N=2|X_1=0) &= P(X_2=1|X_1=0) = P(X_2=1) = p \\ P(N+1=2) &= P(N=1) = P(X_1=1) = p \end{aligned}$$

and

$$\begin{aligned} P(N=3|X_1=0) &= P(X_3=1, X_2=0|X_1=0) = P(X_3=1, X_2=0) = p(1-p) \\ P(N+1=3) &= P(N=2) = P(X_2=1, X_1=0) = p(1-p). \end{aligned}$$

This in turn relies on the fact that the sequence of X_i is i.i.d which is not always the case. See next section for a case when this technique does not apply.

2 Compute the expectation of the first record time

Let X_i be i.i.d. continuous RVs and let $R_i = 1$ if a record happens at time i (that is $X_i = \max(X_1, X_2, \cdots, X_i)$) and 0 otherwise. One can show that the R_i's are independent but they are not identically distributed. In fact, $P(R_i = 1) = 1/i$. Define

$$N = \min\{i \geq 2, R_i = 1\}$$

as the first time a record happens after time 1. Then we know (from Ross problem 2.74) that $E(N) = \infty$. On the other hand, since the R_i's are not iid, imitating the previous example's technique in calculating $E(N)$ conditioning on R_2 may not work. That is

$$\begin{aligned} E(N) = E(E(N|R_2)) &= E(N|R_2 = 0)P(R_2 = 0) + E(N|R_2 = 1)P(R_2 = 1) \\ &= E(N|R_2 = 0)1/2 + 2 \times 1/2. \end{aligned}$$

One may think that $E(N|R_2 = 0) = E(N + 1)$, again motivated by what happened in the previous example. But this is wrong. In fact, the relation

$$N\Big|(R_2 = 0) \stackrel{d}{=} N + 1$$

does not follow. In fact

$$P(N = 3|R_2 = 0) = \frac{P(X_3 > X_2 > X_1)}{P(R_2 = 0)} = \frac{1}{3},$$

or alternatively

$$P(N = 3|R_2 = 0) = P(R_3 = 1|R_2 = 0) = P(R_3 = 1) = \frac{1}{3}$$

while

$$P(N + 1 = 3) = P(N = 2) = P(R_2 = 1) = \frac{1}{2}.$$

Note what fails here is exactly $P(R_3 = 1) \neq P(R_2 = 1)$.

3 Compute the expectation of a compound random variable

Let X_i be iid and N be independent of X_i's. Let

$$S = \sum_{i=1}^{N} X_i.$$

Then S is referred to as *a compound RV.* S can be thought of as the value of a random walk with step sizes X_i that is evaluated at the random time N. To compute $E(S)$ we condition it on N :

$$E(S) = E(E(S|N)) = E(E(\sum_{i=1}^{N} X_i|N)).$$

Now

$$E(\sum_{i=1}^{N} X_i|N) = \sum_{i=1}^{N} E(X_i|N) = \sum_{i=1}^{N} E(X_i) = NE(X_1).$$

Finally

$$E(S) = E(E(S|N)) = E(N)E(X_1).$$

Remark 3.1. Note how in the above computation $E(\sum_{i=1}^{N} X_i|N)$ is a function of N. Also note how $E(X_i|N)$ is reduced to $E(X_i)$ by independence. This is crucical. For example, suppose X_i are iid Bernoulli and we let

$$N = \min\{n \geq 1 : \sum_{i=1}^{n} X_i = 10\}.$$

Then $\sum_{i=1}^{N} X_i = 10$ deterministically and thus

$$E(\sum_{i=1}^{N} X_i|N) = 10 \neq NE(X_1).$$

Finally note how the above result does not change if we weaken the assumption to X_i's being independent with the same mean and not necessarily identically distributed.

4 Compute the expectation of the number of rounds in the hat problem

Suppose in the hat problem the men who picked the right hat leave and the ones who picked the wrong hat throw theirs together again and repeat the process until everyone gets their right hat. Let R_n be the number of rounds it takes in this process for n people. If we denote X_i to be the number of correct hats in round ith then

$$\sum_{i=1}^{R_n} X_i = n.$$

Recall that the expected number of correct hats is 1, irrespective of the total number of hats in the pile. Thus $E(X_i) = 1, \forall i$ and one is tempted to conclude

$$E(\sum_{i=1}^{R_n} X_i) = E(R_n)E(X_1) = n$$

and thus $E(R_n) = 1$. This is wrong, not because the $X_i's$ are not iid (see the remark in the previous section) but because R_n is NOT independent of the X_i. In fact, $R_n = 1$ means $X_1 = n$ and $R_n > 1$ means $X_1 < n$ so they cannot be independent. Nevertheless, it is still reasonable to guess that $E(R_n) = n$.

To derive the result rigorously, we proceed by conditioning on X_1 and using induction on n. That is we note $R_1 = 1$ trivially. Suppose that $E(R_i) = i, i < n$. Now

$$\begin{aligned} E(R_n) = E(E(R_n|X_1)) &= \sum_{i=1}^{n-1} E(R_n|X_1 = i)P(X_1 = i) + E(R_n|X_1 = 0)P(X_1 = 0) \\ &= E(1 + R_{n-i})P(X_1 = i) + E(1 + R_n)P(X_1 = 0). \end{aligned}$$

In the above, we made the assertion that

$$R_n \Big| X_1 \stackrel{d}{=} 1 + R_{n-X_1},$$

but this is true by definition. Using the induction hypothesis that $R_{n-i} = n - i$ for $i \geq 1$ we have

$$\begin{aligned} E(R_n) &= \sum_{i=1}^{n-1}(1 + n - i)P(X_1 = i) + (1 + E(R_n))P(X_1 = 0) \\ &= (1 + n)(1 - P(X_1 = 0)) - E(X_1) + (1 + E(R_n))P(X_1 = 0) \\ &= n(1 - P(X_1 = 0)) + E(R_n)P(X_1 = 0) \end{aligned}$$

where we have used the fact that $\sum_{i=1}^{n-1} iP(X_1 = i) = E(X_1) = 1$. It now follows that $E(R_n) = n$.

5 Compute the expectation for the first time to k consecutive successes

Consider independent Bernoulli X_i with success probability p. Let N_k be the first time that we observe k consecutive successes. Clearly $N_k \geq k$. On the other hand, to compute $E(N_k)$ we must think about the right RV to condition on. While it is true that

$$E(N_k) = E(E(N_k|X_1)),$$

it is not clear what $E(N_k|X_1 = 1)$ is. In fact, we see that the right RV to condition on is N_{k-1}. The conditional relation is

$$\begin{aligned} N_k \Big| N_{k-1} &= N_{k-1} + 1 \text{ with probability } p \\ &= N_{k-1} + 1 + \tilde{N}_k \text{ with probability } 1 - p, \end{aligned}$$

where $\tilde{N}_k$ is a RV with the same distribution with N_k. This comes from considering the result of the trial at the time $N_{k-1} + 1$. Thus

$$\begin{aligned} E(N_k|N_{k-1}) &= p(N_{k-1} + 1) + (1 - p)(N_{k-1} + 1 + E(N_k)) \\ &= 1 + N_{k-1} + (1 - p)E(N_k). \end{aligned}$$

Again note how $E(N_k|N_{k-1})$ is a function of N_{k-1}. Now we have

$$E(N_k) = 1 + E(N_{k-1}) + (1 - p)E(N_k)$$

or

$$E(N_k) = \frac{1 + E(N_{k-1})}{p}.$$

Continue the recurrence with $E(N_1) = \frac{1}{p}$ gives

$$E(N_k) = \sum_{i=1}^{k} \frac{1}{p^i}.$$