Math 244
Fall 2018
Midterm exam 2
4/4/19

Name (Print): ______________________

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	20	
3	15	
4	15	
5	15	
6	20	
Total:	100	

1. (15 points) Consider the equation

$$t^2y'' + 2ty' - 2y = 0, t > 0.$$

It is given that one of the solution is $y_1(t) = t$. Find the general solution to this homogeneous ODE.

Ans: Using the reduction of order method. Let $y_2(t) = ut$. Then

$$\begin{aligned} y_2' &= u + tu' \\ y_2'' &= 2u' + tu''. \end{aligned}$$

Plug in :

$$\begin{aligned} t^2(2u' + tu'') + 2t(u + tu') - 2tu &= 0 \\ t^3u'' + 4t^2u' &= 0 \\ tu'' + 4u' &= 0. \end{aligned}$$

Let $v = u', v' = u''$ then the previous 2nd order ODE becomes

$$\begin{aligned} tv' + 4v &= 0 \\ v' + \frac{4}{t}v &= 0. \end{aligned}$$

The integrating factor is $e^{\int \frac{4}{t}dt} = t^4$. Thus

$$\begin{aligned} (vt^4)' &= 0 \\ v &= \frac{c}{t^4}. \end{aligned}$$

Thus $u = \int u'dt = \int vdt = \frac{c_1}{t^3} + c_2$. We can choose $c_1 = 1, c_2 = 0$ leading to $y_2 = \frac{1}{t^2} = t^{-2}$. The general solution is

$$y(t) = c_1t + c_2t^{-2}.$$

2. (20 points) Consider the equation

$$y'' + 2y' + y = 2e^{-t}.$$

Find the general solution to this equation.

Ans: The homogeneous solution is

$$y_h(t) = c_1e^{-t} + c_2te^{-t}.$$

Thus

$$y_p(t) = At^2e^{-t}.$$

We have

$$\begin{aligned} y_p' &= e^{-t}(2At - At^2) \\ y_p'' &= e^{-t}(At^2 - 4At + 2A). \end{aligned}$$

Plugging in

$$e^{-t}\left[(At^2 - 4At + 2A) + 2(2At - At^2) + At^2\right] = 2e^{-t}.$$

Thus $2A = 2$ or $A = 1$. The general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}.$$

3. (15 points) Consider the equation

$$t^2 y'' - 2y = 3t^2 - 1, t > 0.$$

It is given that the two homogeneous solutions are $y_1(t) = t^2$ and $y_2(t) = t^{-1}$. Find a particular solution to this equation.

Ans: The equation is

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}.$$

We use the formula

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

where

$$\begin{aligned} u_1(t) &= \int \frac{-y_2(t) g(t)}{W[y_1, y_2](t)} dt \\ u_2(t) &= \int \frac{y_1(t) g(t)}{W[y_1, y_2](t)} dt. \end{aligned}$$

Here $g(t) = 3 - \frac{1}{t^2}$ and

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} t^2 & \frac{1}{t} \\ 2t & -\frac{1}{t^2} \end{bmatrix}.$$

Thus $W[y_1, y_2](t) = -1 - 2 = -3$. Computing

$$\begin{aligned} u_1(t) &= -\int \frac{\frac{3}{t} - \frac{1}{t^3}}{-3} dt = \ln(t) + \frac{1}{6t^2} \\ u_2(t) &= \int \frac{t^2 - 1}{-3} dt = \frac{-t^3}{3} + \frac{t}{3}. \end{aligned}$$

Therefore

$$\begin{aligned} y_p(t) &= (\ln(t) + \frac{1}{6t^2})t^2 + (\frac{-t^3}{3} + \frac{t}{3})t^{-1} \\ &= t^2 \ln(t) - \frac{t^2}{3} + \frac{1}{2}. \end{aligned}$$

Since t^2 is a homogeneous solution we can simplify $y_p(t)$ to

$$y_p(t) = t^2 \ln(t) + \frac{1}{2}.$$

4. A spring is stretched 10 cm by a force of 4 N. A mass of 3 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. The mass is pulled down 5 cm below its equilibrium position and given an initial upward velocity of 10 cm/s.

   (a) (5 points) Write down an intial value problem that describes the position y of the mass at any time t

   Ans: We have $k = \frac{4}{.1} = 40, \gamma = \frac{3}{5}$. Thus the IVP is

$$\begin{aligned} 3y'' + \frac{3}{5}y' + 40y &= 0 \\ y(0) = .05, y'(0) &= -0.1. \end{aligned}$$

   (b) (5 points) Find the quasi-frequency of the system.

   We first must verify that the system is underdamped. Indeed, since $\left(\frac{3}{5}\right)^2 - 4 \times 3 \times 40 < 0$ it is underdamped. The quasi-frequency is

$$\omega = \frac{\sqrt{4 \times 3 \times 40 - \left(\frac{3}{5}\right)^2}}{2 \times 3}.$$

   (c) (5 points) Does the mass crosses the equilibrium position more than one time? Explain.

   Ans: Because the system is underdamped, it will cross the equilibrium position more than one time.

5. (15 points) Consider the equation

$$y''' - 2y'' + y' = t^3 + 2e^t.$$

Determine a suitable form of $y_p(t)$, a particular solution to this problem. Do NOT evaluate the constants of the form.

Ans: The characteristic equation is $r^3 - 2r^2 + r = r(r-1)^2 = 0$. Thus the homogeneous solution is

$$y_h(t) = c_1 + c_2 e^t + c_3 t e^t.$$

Thus a suitable form of $y_p(t)$ is

$$y_p(t) = t(At^3 + Bt^2 + Ct + D) + Et^2 e^t.$$

6. (20 points) Consider the system :

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}.$$

Find the general solution to this system.

Ans: We have

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} & (1-\lambda)[(\lambda+1)(\lambda-2)+1] - 3(1+\lambda) + 2 + 4(3-2(2-\lambda) \\ = \ & (1-\lambda)[(\lambda+1)(\lambda-2)+1] + 5\lambda - 5 \\ = \ & (1-\lambda)[\lambda^2 - \lambda - 6] \\ = \ & (1-\lambda)(\lambda-3)(\lambda+2). \end{aligned}$$

For $\lambda = 1$:

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}.$$

For $\lambda = -2$:

$$\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

For $\lambda = 3$:

$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus

$$\mathbf{x}(t) = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t}.$$