**Math 244**
**Fall 2018**
**Midterm exam 1**
**2/21/19**

**Name (Print):** ______________________

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This exam contains 5 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

| Problem | Points | Score |
| --- | --- | --- |
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 15 | |
| Total: | 100 | |

1. (15 points) Solve the IVP

$$\begin{aligned} 16y'' + 24y' + 9y &= 0 \\ y(0) = y'(0) &= 1 \end{aligned}$$

Ans: The characteristic equation is

$$\begin{aligned} 16r^2 + 24r + 0 &= 0 \\ (4r+3)^2 &= 0. \end{aligned}$$

So the general solution is $y(t) = c_1 e^{-\frac{3}{4}t} + c_2 t e^{-\frac{3}{4}t}$. Plug in the initial conditions:

$$\begin{aligned} c_1 &= 1 \\ -\frac{3}{4}c_1 + c_2 &= 1. \end{aligned}$$

Thus $c_1 = 1, c_2 = \frac{7}{4}$.

2. A tank initially (at time $t = 0$) contains 100 gal of pure water. Water contaning a salt concentration of $e^{-2t}$ oz/gal flows into the tank at a rate of 2 gal/min, and the (well mixed) mixture in the tank flows out at the same rate.

   (a) (10 points) Find the amount of salt in the tank at any time.
   Let $Q(t)$ be the amount of salt in the tank at time $t$. Then

$$\begin{aligned} Q'(t) &= 2e^{-2t} - 2\frac{Q}{100} \\ Q(0) &= 0. \end{aligned}$$

   That is

$$\begin{aligned} Q'(t) + \frac{1}{50}Q &= 2e^{-2t} \\ Q(0) &= 0. \end{aligned}$$

   Using the integrating factor $e^{\frac{t}{50}}$ gives

$$(Qe^{\frac{t}{50}})' = 2e^{-\frac{99t}{50}}.$$

   Thus

$$Q(t) = -\frac{100}{99}e^{-2t} + ce^{-\frac{t}{50}}.$$

   Plug in the initial condition $Q(0) = -\frac{100}{99} + c = 0$. Thus $Q(t) = -\frac{100}{99}e^{-2t} + \frac{100}{99}e^{-\frac{t}{50}}$.

   (b) (5 points) Find the limiting amount of salt in the tank as $t \to \infty$.
   As $t \to \infty, Q(t) \to 0$.

3. (15 points) Solve the IVP

$$\begin{aligned} e^t y' + y &= 1 \\ y(0) &= 1 \end{aligned}$$

We rewrite it as

$$\begin{aligned} y' + e^{-t}y &= e^{-t} \\ y(0) &= 1 \end{aligned}$$

Using the integrating factor $e^{-e^{-t}}$ we have

$$(ye^{-e^{-t}})' = e^{-t}e^{-e^{-t}}.$$

Use the substitution $u = -e^{-t}$ we have

$$\int e^{-t}e^{-e^{-t}}dt = \int e^u du = e^u + c.$$

Thus

$$\begin{aligned} ye^{-e^{-t}} &= e^{-e^{-t}} + c \\ y(t) &= 1 + ce^{e^{-t}}. \end{aligned}$$

Plug in the initial condition $y(0) = 1 + ce = 1$. Thus $c = 0$. Thus $y(t) = 1$.

4. Consider the IVP

$$\begin{aligned} \frac{dy}{dt} &= y - t \\ y(0) &= 1. \end{aligned}$$

(a) (5 points) Find the approximate value of the solution to the IVP at $t = 2$ using Euler method with step size $h = 0.5$.

$$\begin{aligned} y(0.5) &\approx y(0) + y'(0)0.5 = 1 + (1-0)0.5 = 1.5 \\ y(1) &\approx y(0.5) + y'(0.5)0.5 = 1.5 + (1.5 - .5)0.5 = 2 \\ y(1.5) &\approx y(1) + y'(1)0.5 = 2 + (2-1)0.5 = 2.5 \\ y(1) &\approx y(1.5) + y'(1.5)0.5 = 1.5 + (2.5 - 1.5)0.5 = 3. \end{aligned}$$

(b) (5 points) Find the exact value of the solution to the IVP at $t = 2$. We have

$$y' - y = -t.$$

Using integrating factor $e^{-t}$ gives

$$(ye^{-t})' = -te^{-t}.$$

Thus

$$ye^{-t} = te^{-t} + e^{-t} + c.$$

Or

$$y(t) = t + 1 + ce^t.$$

Plug in $y(0) = 1$ gives $c = 0$. Thus $y(2) = 2 + 1 = 3$.

5. (15 points) Solve the IVP

$$\begin{aligned} 6y'' - y' - y &= 0 \\ y(0) = y'(0) &= 1 \end{aligned}$$

The characteristic equation is

$$6r^2 - r - 1 = 0$$

which has solutions $r = 1/2, r = -1/3$. Thus the general solution is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-\frac{t}{3}}.$$

Plug in the initial conditions:

$$\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= \frac{c_1}{2} - \frac{c_2}{3} = 1. \end{aligned}$$

We have $c_1 = \frac{8}{5}, c_2 = -\frac{3}{5}$.

6. (15 points) Solve the IVP

$$\begin{aligned} y' &= te^{y-t} \\ y(0) &= 2. \end{aligned}$$

Ans: By seperation of variables:

$$\int e^{-y} dy = \int te^{-t} dt.$$

Thus

$$-e^{-y} = -te^{-t} - e^{-t} + c.$$

Plug in the initial condition:

$$-e^{-2} = -1 + c.$$

Thus $c = 1 - e^{-2}$ and

$$-e^{-y} = -te^{-t} - e^{-t} + 1 - e^{-2}.$$

Thus last step is not required but we also have

$$y = -\ln\left(te^{-t} + e^{-t} - 1 + e^{-2}\right), t \geq 0.$$

7. (15 points) Solve the IVP

$$\begin{aligned} y'' + 2y' + 1.25y &= 0 \\ y(0) = y'(0) &= 1 \end{aligned}$$

The characteristic equation is

$$r^2 + 2r + 1.25 = 0.$$

That is

$$(r+1)^2 = -\frac{1}{4}.$$

Thus $r = -1 \pm \frac{1}{2}i$. Thus the general solution is

$$y(t) = c_1 e^{-t}\cos(\frac{t}{2}) + c_2 e^{-t}\sin(\frac{t}{2}).$$

Plug in the initial conditions:

$$\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= -c_1 + \frac{c_2}{2} = 1. \end{aligned}$$

Thus $c_1 = 1, c_2 = 4$.