Math 244
Spring 2019
Final exam
5/10/19

Name (Print): ____________________________

This exam contains 8 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- In any problem that you want to use the method of variation of parameters, remember that the leading coefficient (of y'') must be 1 before the formula can be applied.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	15	
5	20	
6	15	
7	15	
8	15	
9	15	
10	15	
11	15	
12	15	
Total:	200	

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

1. (20 points) Consider the system :

$$\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{x}.$$

Find the general solution to this system.

Ans: The eigenvalues satisfy $(\lambda - 1)(3 + \lambda) + 5 = 0$. That is $\lambda^2 + 2\lambda + 2 = 0$. Thus $\lambda = -1 \pm i$. If $\lambda = -1 + i$ then

$$A - \lambda I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}.$$

Thus a corresponding eigenvector is

$$\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i.$$

Thus the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) + c_2 e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right) \\ &= e^{-t}(c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}). \end{aligned}$$

2. (20 points) Let

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}.$$

Find e^A.

The eigenvalues satisfy $(\lambda-5)(\lambda-1)+3=0$. That is $\lambda^2-6\lambda+8=0$. Thus $\lambda=2,4$. If $\lambda=2$

$$A-\lambda I=\begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix}.$$

Thus a corresponding eigenvector is $\begin{bmatrix}1\\ 3\end{bmatrix}$. If $\lambda=4$

$$A-\lambda I=\begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix}.$$

Thus a corresponding eigenvector is $\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus

$$P=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}, P^{-1}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}.$$

And

$$e^A=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}e^2 & 0\\ 0 & e^4\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}.$$

3. (20 points) Consider the nonhomogeneous system

$$\mathbf{x}'=\begin{bmatrix}5 & -1\\ 3 & 1\end{bmatrix}\mathbf{x}+\begin{bmatrix}\cos(t)\\ e^t\end{bmatrix}.$$

Find the corresponding diagonalized nonhomogeneous system. Do not solve the diagonalized system.

Using the answer from the previous question

$$\mathbf{h}(t)=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}\cos(t)\\ e^t\end{bmatrix}=\begin{bmatrix}\frac{e^t-\cos t}{2}\\ \frac{3\cos t-e^t}{2}\end{bmatrix}.$$

The diagonalized system is

$$\mathbf{y}'=\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}\mathbf{y}+\begin{bmatrix}\frac{e^t-\cos t}{2}\\ \frac{3\cos t-e^t}{2}\end{bmatrix}.$$

4. (15 points) Consider the equation

$$t^2y''-t(t+2)y'+(t+2)y=2t^3, t>0.$$

It is given that the two homogeneous solutions are $y_1(t)=t$ and $y_2(t)=te^t$. Find a particular solution to this equation.

The equation is

$$y''-\frac{t+2}{t}y'+\frac{t+2}{t^2}y=2t, t>0.$$

We use the formula

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$\begin{aligned} u_1(t) &= \int \frac{-y_2(t)g(t)}{W[y_1,y_2](t)}dt \\ u_2(t) &= \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt. \end{aligned}$$

Here $g(t) = 2t$ and

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} t & te^t \\ 1 & e^t(t+1) \end{bmatrix}.$$

Thus $W[y_1, y_2](t) = t^2e^t$. Computing

$$\begin{aligned} u_1(t) &= -\int \frac{2t^2e^t}{t^2e^t}dt = -2t \\ u_2(t) &= \int \frac{2t^2}{t^2e^t}dt = -2e^{-t}. \end{aligned}$$

Therefore

$$y_p(t) = -2t^2 - 2t.$$

Because t is a homogeneous solution, we can simplify to

$$y_p(t) = -2t^2.$$

5. (20 points) Consider the system :

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} \mathbf{x}.$$

Find the general solution to this system.

The eigenvalues are $\lambda = 1, 1, 2$. If $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix}.$$

Thus an associated eigenvector is $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$. The pseudo eigenvector associated $\mathbf{u}_1$ satisfies

$$\begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix} \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}.$$

Thus a choice for $\mathbf{u}_2$ is

$$\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{4} \\ 0 \\ \frac{21}{4} \end{bmatrix}.$$

If $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{bmatrix}.$$

Thus an associated eigenvector is $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} te^t + \begin{bmatrix} -\frac{1}{4} \\ 0 \\ \frac{21}{4} \end{bmatrix} e^t \right) \\ &+ c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}. \end{aligned}$$

6. Consider the nonlinear system:

$$\begin{aligned} \frac{dx}{dt} &= x + x^2 + y^2 \\ \frac{dy}{dt} &= y - xy. \end{aligned}$$

(a) (5 points) Determine all critical points of the system.
The two systems for critial points are

$$\begin{aligned} x + x^2 + y^2 &= 0 \\ y &= 0 \end{aligned}$$

and

$$\begin{aligned} x + x^2 + y^2 &= 0 \\ 1 - x &= 0. \end{aligned}$$

The first has two solutions: $(0,0), (-1,0)$. The second has no real solution for y.

(b) (5 points) Find the approximating linear system near each critical point from part a.

$$\begin{aligned} f_x &= 1+2x \\ f_y &= 2y \\ g_x &= -y \\ g_y &= 1-x. \end{aligned}$$

Thus the two linearized systems are : At $(0,0)$

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}.$$

At $(-1,0)$

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} (\mathbf{x} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}).$$

(c) (5 points) Classify each critical point from part a.

At $(0,0)$ since the eigenvalues are both 1, it is either a node or a spiral point and is unstable.

At $(-1,0)$, since $-1 = \lambda_1 < 0 < 2 = \lambda_2$ it is an unstable saddle point.

7. Consider the following system describing the behavior of two species:

$$\begin{aligned} \frac{dx}{dt} &= x(1.5 - x - 0.5y) \\ \frac{dy}{dt} &= y(2 - 0.5y - 1.5x). \end{aligned}$$

(a) (5 points) Is this a competing species or predator-prey system?

This is competing species because the presence of one has a negative effect (negative derivative with large enough population) on the other.

(b) (10 points) Describe the limiting behavior of the population as $t \to \infty$.

As $\alpha_1\alpha_2 = 0.5 < \sigma_1\sigma_2 = 0.75$, one population will die out in the long run. Alternatively, one can check that the critical point (1, 1) is an unstable saddle point.

8. Consider the following system describing the behavior of two species:

$$\begin{aligned} \frac{dx}{dt} &= x(1.5 - 0.5y) \\ \frac{dy}{dt} &= y(-0.5 + x). \end{aligned}$$

(a) (5 points) Is this a competing species or predator-prey system?

This is a predator prey system : for large enough y, $\frac{dx}{dt} < 0$ and for large enough x, $\frac{dy}{dt} > 0$. Thus x is the prey and y the predator.

(b) (10 points) Describe the limiting behavior of the population as $t \to \infty$.

The system will enter a cyclical pattern in the long run.

9. (15 points) Consider the equation

$$y^{(4)} - y = 3t^2e^t + 4t\cos(t) - \sin(2t).$$

Determine a suitable form of $y_p(t)$, a particular solution to this problem. Do NOT evaluate the constants of the form.

The homogeneous solution is

$$y_h(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t.$$

Thus a suitable form of $y_p(t)$ is

$$y_p(t) = t(At^2 + Bt + C)e^t + t(Dt + E)\cos t + t(Ft + G)\sin t + H\sin 2t + I\cos 2t.$$

10. (15 points) Find the general solution to the equation

$$y'' - 2y' + y = e^t + 4.$$

The homogeneous solution is

$$y_h(t) = c_1e^t + c_2te^t.$$

The form of a particular solution is

$$y_p(t) = At^2e^t + B.$$

Plugging in we get

$$[A(t^2 + 4t + 2) - 2A(t^2 + 2t) + At^2]e^t + B = e^t + 4.$$

Thus $A = 1/2, B = 4$. The general solutionis

$$y(t) = c_1e^t + c_2te^t + \frac{1}{2}t^2e^t + 4.$$

11. (15 points) Find the solution to the given initial value problem

$$y' - 2y = 2te^{2t}, y(0) = 1.$$

The integrating factor is e^{-2t}. Thus we have

$$(ye^{-2t})' = 2t.$$

Thus

$$y(t) = (t^2 + c)e^{2t}.$$

The initial condition gives $c = 1$.

12. (15 points) Find the solution to the given initial value problem in explicit form (i.e. y as a

function of x)

$$y' = \frac{1-2x}{y}, y(1) = -2.$$

Separating variables we have

$$ydy = (1-2x)dx.$$

Thus

$$\frac{y^2}{2} = x - x^2 + c.$$

We have

$$y(t) = \pm\sqrt{2(x - x^2 + c)}.$$

Plug in the initial condition gives $c = 2$ and

$$y(t) = -\sqrt{2(-x^2 + x + 2)}.$$