Math 622
Spring 2017
Midterm exam 2
4/12/17

Name (Print): ____________________

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 2 pages of notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider a compound Poisson process $Q(t) = \sum_{i=1}^{N(t)} Y_i$ where Y_i are i.i.d. discrete random variables with distribution

$$P(Y_i = 1) = P(Y_i = -1) = 1/4, P(Y_i = 2) = 1/2.$$

Consider a perpetual game based on $Q(t)$ as followed: each time $Q(t)$ jumps up with size k the player receives k dollar(s) and has the option to continue playing or stop and receive the winning (which is cumulative). If $Q(t)$ jumps down the player has to stop and receive 0 dollar. If the cumulative winning is 5 or above the player also has to stop and receive 0 dollar. What is the risk neutral price to play this game?

Ans: Let $V(n)$ be the value of the game when the player has won n dollars. Then $V(n) = 0, n \geq 5$. We look for $V(0)$. From any state n, the possible next states are $n-1, n+1, n+2$. Thus we see that $V(4) = 4$ since at state 4 no matter what the next outcome the player will lose his fortune. The continuation value at $n = 3$ is $1/4V(4) = 1$. Thus the player will stop at 3 as well and $v(3) = 3$. The continuation value at $n = 2$ is $1/4V(3) + 1/2(2+2) = 2 + 3/4$. Thus the player will continue to play at $n = 2$ and $V(2) = 2.75$. The continuation value at $n = 1$ is $1/4V(2) + 1/2V(3) = 1/4(2.75) + 1.5 = 2.1875$. Thus the player will continute to play at $n = 1$ and $V(1) = 2.1875$. Finally the player will continute to play at $n = 0$. The value of the game, which is $V(0)$ is $1/4V(1) + 1/2V(2)$ where we found $V(1), V(2)$ above.

2. (20 points) Let $X(t)$ be the solution to the following equation

$$\begin{aligned} dX(t) &= 3dt + X(t-)dN(t), \\ X(0) &= 1, \end{aligned}$$

where $N(t)$ is a Poisson process with rate 5. Find $E(X(2))$. (Hint : Writing an equation that $E(X(t)$ satisfies might be a more convenient approach than solving for $X(t)$ directly).

Ans:

$$\begin{aligned} X(t) &= 1 + 3t + 5\int_0^t X(s-)ds + \int_0^t X(s-)d(N(s) - 5s) \\ &= 1 + 3t + 5\int_0^t X(s)ds + \int_0^t X(s-)d(N(s) - 5s). \end{aligned}$$

Taking expectation on both sides, using the fact that $\int_0^t X(u-)d(N(u) - \lambda u)$ is a martingale gives us

$$E(X(t)) = 1 + 3t + 5\int_0^t E(X(s))ds.$$

Denoting $f(t) := E(X(t)$ we see that $f(t)$ satisfies the ODE

$$\begin{aligned} f'(t) &= 3 + 5f(t) \\ f(0) &= 1. \end{aligned}$$

That is $[e^{-5t}f(t)]' = 3e^{-5t}$. The solution is

$$f(t) = -\frac{3}{5} + \frac{8}{5}e^{5t}.$$

Thus $E(X(2)) = f(2) = -\frac{3}{5} + \frac{8}{5}e^{10}$.

3. (20 points) Consider a 3 asset model:

$$\begin{aligned}
dS^1(t) &= \mu S^1(t)dt + S^1(t)dW^1(t) + S^1(t)dW^2(t) \\
dS^2(t) &= \mu S^2(t)dt + S^2(t)dW^1(t) + S^2(t-)d(N(t) - \lambda t) \\
dS^3(t) &= \mu S^3(t)dt + S^3(t)dW^2(t) + S^3(t-)d(N(t) - \lambda t),
\end{aligned}$$

where W^1, W^2 are independent Brownian motions and $N(t)$ a Poisson process with rate λ. Assume the risk-free rate is $r > 0$ and λ much bigger than μ, r. Is there a risk neutral probability for this model? If so, is it unique?

Ans: Denoting $\tilde{W}^1(t) = W^1(t) + \theta_1 t, \tilde{W}^2(t) = W^2(t) + \theta_2 t, \tilde{\lambda}$ the new rate of $N(t)$ we have the market price of risk equation to be

$$\begin{aligned}
r + \theta_1 + \theta_2 &= \mu \\
r + \theta_1 - \tilde{\lambda} &= \mu - \lambda \\
r + \theta_2 - \tilde{\lambda} &= \mu - \lambda.
\end{aligned}$$

The las two equations imply $\theta_1 = \theta_2$ and the first equation implies $\theta 1 = \theta_2 = \frac{\mu - r}{2}$. There exists a $\tilde{\lambda} > 0$ if $\tilde{\lambda} = \lambda - \frac{\mu - r}{2} > 0$. Since λ is much bigger than μ, r by assumption this is true and the model has a unique risk neutral probability.

4. (20 points) Consider an asset model under the physical probability

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) - S(t-)dQ(t), \\ S(0) &= 50, \end{aligned}$$

where $Q(t)$ is a compound Poisson process with $Q(t) = \sum_{i=1}^{N(t)} Y_i$ and $Y(i)$ are i.i.d. $N(.25, (.05)^2)$. Again here $Y(i)$ captures the percentage of the change in S_t value. Suppose $N(t)$ is a Possion process with rate $\lambda = \frac{1}{10}$. All time units are in years. Suppose a credit default event is when the stock value falls in 40 % or more. Describe how we can efficiently compute via simulation the probability that the company has a credit default event triggered during the first year. (Note that both probabiliy of a jump happening in the first year and a particular jump of value 40 % or more are rare events).

Ans: Since N_t and Y_i are independent we can do the following: Compute the probabilities $p_k = \frac{\lambda^k e^{-\lambda}}{k!}$ that $N(t)$ jumps k times during the interval $[0,1]$ analytically. Next we can simulate the probability $P(Y_1 > 0.4)$ using importance sampling technique as mentioned before. The probability we are interested in can be approximated by $\sum_{k=1}^{N} p_k P(Y_1 > 0.4)^k$ for a large N.

Alternatively, we can use the change of measure formulation for compound Poisson process with continuous jump distribution (discussed in the lecture note and the textbook) to change the rate of $N(t)$ and the distribution of Y_i to a more suitable choice at teh same time (which could be making λ higher, e.g. 1, and Y_i having $N(\mu, (0.5)^2)$ distribution where $\mu = 0.40$. Once under the new measure we can simulate Q_t using the usual approach.

5. (20 points) Brownian bridge : A popular way to simulate Brownian motion path is via the Brownian bridge. At the heart of it is the following result: for times $t_1 < t < t_2$ the distribution of $W(t)$ conditioned on $W(t_1) = a, W(t_2) = b$ is Normal with mean $\frac{(t_2-t)a+(t-t_1)b}{t_2-t_1}$ and variance $\frac{(t-t_1)(t_2-t)}{t_2-t_1}$.

   a) Utilizing the fact that for a Brownian motion W and $s < t, W_t - W_s$ is indpendent of W_s what is the distribution of W_t conditioned on $W_s = a$? Explain how your finding is consistent with the above result about the Brownian bridge.

   Ans: $W_t|W_s = a$ has $N(a, t-s)$ distribution. It is because $W_t = W_s + W_t - W_s$ and $W_t - W_s$ has $N(0, t-s)$ distribution. Using the independence between $W_t - W_s$ and W_s we get the result. This is consistent with the Brownian bridge result if we let $t_1 = s, t_2 \to \infty$ in the Brownian bridge result to get the mean is

$$\lim_{t_2\to\infty} \frac{(t_2 - t)a + (t - s)b}{t_2 - s} = a$$

   and the variance is

$$\lim_{t_2\to\infty} \frac{(t - s)(t_2 - s)}{t_2 - s} = t - s.$$

   b) Again using the Brownian bridge rseult, for $s < t$ what is the distribution of W_s conditioned on $W_t = b$?

   Ans: Setting $t_1 = 0, a = 0, t = t_2$ in the Brownian bridge result gives $W_s|W_t = b$ has $N(\frac{(s-t)b}{t}, \frac{s(t-s)}{t})$ distribution.

c) Suppose we have already generated a Brownian sample path at sample times $0 = t_0 < t_1 < \cdots t_n = T$ (that is at n time points). Explain how we can use the Brownian bridge result to efficiently generate a Brownian sample paths at (roughly) $2n$ time points $0 = t_0 < \tilde{t}_0 < t_1 < \tilde{t}_1 < t_2 < \cdots < \tilde{t}_{n-1} < t_n = T$ where $\tilde{t}_i$ are the new sample points in between the old sample points t_i.

Ans: We can use the result of the Brownian bridge to sample the $W_{\tilde{t}_i}$ points given $W_{t_i}, W_{t_{i+1}}$ from the old path. This gives a new path with n extra data points.

Scratch (Won't be graded)

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