Math 622
Spring 2017
Midterm exam 1
3/1/16

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 2 pages of notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let $\tilde{W}_t$ be a given Brownian motion. Let r_t follow a CIR model:

$$dr_t = k(\mu - r_t)dt + \sigma_1 \sqrt{r_t} d\tilde{W}_t$$

and S_t follow a Geometric Brownian motion model :

$$dS_t = r_t S_t dt + \sigma_2 S_t d\tilde{W}_t.$$

Consider the European call option with strike K and expiry T: $V_T = (S_T - K)^+$. Find the PDE representation for the value of this call option at any time $t, 0 \leq t \leq T$. Make sure to provide all necessary boundary conditions.

Ans: From the Markov properties of r_t, S_t there exists $v(t, x, y)$ so that $v(t, r_t, S_t) = V_t$. Thus the martingale condition of $e^{-\int_0^t r_s ds} V_t$ requires:

$$e^{-\int_0^t r_s ds}(-r_t v + v_t + v_x r_t S_t + \frac{1}{2} v_{xx} (\sigma_2)^2 S_t^2 + \frac{1}{2} v_{yy} (\sigma_1)^2 r_t + v_{xy} \sigma_1 \sigma_2 \sqrt{r_t} S_t) = 0.$$

That is

$$-yv + v_t + v_x yx + \frac{1}{2} v_{xx} (\sigma_2)^2 x^2 + \frac{1}{2} v_{yy} (\sigma_1)^2 y + v_{xy} \sigma_1 \sigma_2 \sqrt{y} x = 0.$$

The natural range (t, x, y) is $[0, T] \times [0, \infty) \times [0, \infty)$. The boundary conditions are

$$\begin{aligned} v(T, x, y) &= (x - K)^+ \\ v(t, 0, y) &= 0 \\ v(t, x, \infty) &= 0. \end{aligned}$$

2. (20 points) Let S_t follow a Black-Scholes model

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\tilde{W}_t, \\ S_0 &= x. \end{aligned}$$

Let B, L be two constants such that $B < x < L$. Consider the perpetual two barrier option that pays B if the discounted stock value, that is $e^{-rt}S_t$ hits B before L and pays L if the discounted stock value $e^{-rt}S_t$ hits L before B. Find the value of this option at time 0. (Hint: Use optional sampling theorem. You can use without proving the fact that the first time $e^{-rt}S_t$ hits either level B or L is bounded).

Ans: Denoting $M_t = e^{-rt}S_t$ (noting M_t is a martingale) and let $\tau = inf(t \geq 0, M_t = B \text{ or } L)$. Then τ is bounded and

$$x = \tilde{E}(M_\tau)$$

by optional sampling. Expanding this expression gives:

$$x = L\tilde{P}(M_\tau = L) + B\tilde{P}(M_\tau = B).$$

Coupling with the fact that $\tilde{P}(M_\tau = L) + \tilde{P}(M_\tau = B) = 1$ gives

$$\tilde{P}(M_\tau = L) = \frac{x - B}{L - B}$$

$$\tilde{P}(M_\tau = B) = \frac{L - x}{L - B}.$$

Thus by risk neutral pricing, the option price is

$$V_0 = \tilde{E}(e^{-r\tau}[L\mathbf{1}_{M_\tau = L} + B\mathbf{1}_{M_\tau = L}])$$

This expression is hard to compute if $r > 0$. On the other hand, if $r = 0$ and the option pays a if S_t hits L before B and pays b if S_t hits B before L then

$$V_0 = a\frac{x - B}{L - B} + b\frac{L - x}{L - B}.$$

3. (20 points) a) Consider a sequence of numbers $x_1, x_2, \cdots, x_n$ that lie in the interval $[0,1]$. Suppose we want to test whether such sequence comes from a Uniform[0,1] distribution. Describe a reasonable procedure that would accomplish this.

   Ans: We can partition the interval $[0,1]$ into N equally-spaced subintervals. If x_i are uniformly distributed then on average there are n/N values of x_i in each subinterval. Let n_k be the number of x_i falling into the k-subinterval. Then we accept xi to be uniformly distributed if $\sum_k (n_k - n/N)^2$ is small and reject otherwise. The threshold for rejection depends on the confidence level of choice.

   b) Suppose we have a Uniform[0,1] random generator. Describe how we can simulate the throw of a unfair die X with the following distribution:

$$P(X=1) = 1/2; P(X=2) = 1/4; P(X=3) = 1/8; P(X=4) = 1/16; P(X=5) = P(X=6) = 1/32.$$

   Ans: Use the random generator to generate a sequence $x_1, x_2, \cdots, x_n$ in $[0,1]$. Define a map $y_k = f(x_k), k = 1, \cdots, n$ as followed:

$$\begin{aligned}
y_k &= 1 \text{ if } 0 \leq x_k < 1/2 \\
y_k &= 2 \text{ if } 1/2 \leq x_k < 3/4 \\
y_k &= 3 \text{ if } 3/4 \leq x_k < 7/8 \\
y_k &= 4 \text{ if } 7/8 \leq x_k < 15/16 \\
y_k &= 5 \text{ if } 15/16 \leq x_k < 31/32 \\
y_k &= 6 \text{ if } 31/32 \leq x_k < 1.
\end{aligned}$$

4. (20 points) Let S_t follows the Black-Scholes mode:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

where

$$S_0 = 54.99, \sigma = 0.2, r = 0.05.$$

Consider the up and out call option

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{0\le u\le T} S_u \le L\}},$$

where $L = 55, K = 45, T = 2$.

We want to compute the probability that the option expires in the money. Suggest an efficient algorithm to do so. Write down the steps or pseudo code of your algorithm (no need to be explicit about how you generate the path etc. The focus is on how you solve the efficiency problem).

Ans: The usual Monte-Carlo will generate many paths expiring out of the money ($r - \frac{1}{2}\sigma^2 > 0$ also add to the up-drift). The probability of expiring in the money should be quite small (close to 0, but not exactly 0). To make this algorithm more efficient, we can change the drift of S_t to a large negative number using Girsanov theorem:

$$dS_t = \alpha S_t dt + \sigma S_t d\hat{W}_t,$$

where α is a large negative number of choice and $\hat{W}_t = \tilde{W}_t + \frac{r-\alpha}{\sigma} d_t$. Denoting $m = \frac{r-\alpha}{\sigma}$ the Girsanov change of measure kernel is

$$Z_t = e^{-\frac{1}{2}m^2 t - m\tilde{W}_t} = e^{\frac{1}{2}m^2 t - m\hat{W}_t}.$$

Then

$$\tilde{E}(\mathbf{1}_{\{\max_{0\le u\le T} S_u \le L\}} \mathbf{1}_{\{S_T \ge K\}}) = \hat{E}(\frac{1}{Z_T} \mathbf{1}_{\{\max_{0\le u\le T} S_u \le L\}} \mathbf{1}_{\{S_T \ge K\}}).$$

Under $\hat{P}, S_t$ has a negative drift α and we have expressed Z_T in terms of the $\hat{W}_T$ Brownian motion. Thus we should be able to generate more paths that expire in the money under $\hat{P}$. Note: α should not be chosen too large as it may also make S_T expiring out of the money being lower than K.

5. (20 points) A popular way to generate independent Normal random variables is via Box-Muller transformation. Specifically, let U_1, U_2 be two independent Uniform[0,1] random variables. Denote

$$\begin{aligned} Z_1 &= \sqrt{-2\log U_1}\cos(2\pi U_2) \\ Z_2 &= \sqrt{-2\log U_1}\sin(2\pi U_2). \end{aligned}$$

Then (Z_1, Z_2) are independent standard Normal random variables. Show that Box-Muller works. That is show (Z_1, Z_2) have the described distribution. (Hint: Use polar coordinates representation for joint standard Normal distribution).

Ans: We want to show that Z_1, Z_2 has joint independent standard Normal density:

$$f_{Z_1Z_2}(z_1, z_2) = \frac{1}{2\pi}e^{-\frac{z_1^2+z_2^2}{2}}dz_1dz_2.$$

Under polar coordinate : $z_1 = r\cos(\theta), z_2 = r\sin(\theta)$. Thus we ant to show the above density as

$$f_{R\Theta}(r, \theta) = \frac{1}{2\pi}e^{-\frac{r^2}{2}}rdrd\theta. \tag{1}$$

Observe that $R = \sqrt{-2\log U_1}$ and $\Theta = 2\pi U_2$ in the Box-Muller representation. Now

$$\begin{aligned} P(R < r) = P(U_1 > e^{-\frac{r^2}{2}}) = 1 - e^{-\frac{r^2}{2}} \\ P(\Theta < \theta) = P(2\pi U_2 < \theta) = \frac{\theta}{2\pi}. \end{aligned}$$

Thus

$$\begin{aligned} f_R r &= re^{-\frac{r^2}{2}} \\ f_\Theta(\theta) &= \frac{1}{2\pi}. \end{aligned}$$

By independence of U_1, U_2, R, Θ are also independent. Thus their joint density is the product of their individual densities, which is of the form (1) above.

Scratch (Won't be graded)

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