Homework 5 (Due 2/24/2017)

Math 622

March 3, 2017

In this homework there will be some Monte-Carlo simulation and numerical integration. Report all your codes with the homework.

1. Consider the Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

where

$$S_0 = 50, \sigma = 0.2, r = 0.01.$$

a) Consider a up and out barrier call option with strike $K = 50$ and barrier $B = 55$, expiry $T = 1$. Use Monte-Carlo simulation to find V_0.

b) Use formula (7.2.5) in Shreve and numerical integration (double integration) to find V_0.

c) Which method (a or b) do you think is more efficient? (Of course it depends on how precise the convergence in each method is, choose a reasonable same range of convergence for your comparison.)

2. Let X be a random variable and $F_X(x) := P(X \leq x)$ be its cumulative distribution function. F_X is non decreasing in general. For simplicity let's suppose here F_X is strictly increasing. Since F_X is monontone there exists an inverse function F_X^{-1}. Let U be a uniform $[0, 1]$ distribution and let $\tilde{X} := F_X^{-1}(U)$. Show that $\tilde{X}$ has the same distribution as X. This is the simplest way to sample from the X-distribution.

3. Use the method from part 2) to generate 10,000 Exponential (1) random variables. Check your answer by providing an estimate for $P(X \leq 2)$, where X has Exponential (1) distribution using your 10,000 sample points. Report the error between this estimate and the analytical answer from explicit integration.

4. Rejection sampling: In many cases, $F_X(x)$ and $F_X^{-1}(x)$ are not easy to find explicitly (e.g. Normal distribution). Thus we need another method to sample from a distribution X knowing its density. The procedure is as described in class:

i) Find a density function $g(x)$ that we can sample from. The condition that $g(x)$ must satisfy is $f(x) \leq cg(x)$ for some $c \geq 1$.

ii) Generate X from density $g(x)$. Generate a U uniform $[0, 1]$ independent of X.

(iii) (Acceptance / Rejection) If $U \leq \frac{f(X)}{cg(X)}$ we accept X into the sample. If not we reject this value and repeat step (ii) until we find a value X that we accept.

The basis for this sampling is the following: Suppose X has density $g(x)$. Let $U, f(x), c$ be as above. Show that

a)

$$P(U \leq \frac{f(X)}{cg(X)}) = \frac{1}{c}$$

.

b) Given a set A, show that

$$P(X \in A | U \leq \frac{f(X)}{cg(X)}) = \int_A f(x)dx.$$

Thus conditioned on the fact that we accept, X is distributed as if it was sampled from the f density.

5. In this problem we want to use rejection sampling to compute $E(M_t | W_t = 0)$ for $t = 1$. The density of $M_t | W_t$ is given in Shreve's Corollary 3.7.4.

a) Use numerical integration to compute $E(M_t | W_t = 0)$.

b) Use rejection sampling to compute $E(M_t | W_t = 0)$. For $g(x)$ use Exponential (1) distribution. You need to choose c carefully (the larger the c the more time you will reject hence the less efficient the sampling). What value of c do you use?

c) Compare the two methods. Which one do you think is more efficient?

6. Extra credit (5 points in the first midterm - must be submitted before midterm 1)

It is reasonable to believe that in computing $E(M_t)$ for large t (say $t = 1000$) rejection sampling maybe more efficient than simple Monte Carlo since we avoid generating the whole path of the Brownian motion. The difficulty with computing $E(M_t)$ using rejection sampling is for different $W_t = w$ we need to estimate a different c_w to make sure the density $f_{M_t|W_t=w} \leq c_w g(x)$. This difficulty is addressed under the (improved) version of rejection sampling called Metropolis-Hasting Acceptance-Rejection sampling. This algorithm allows us to pick a c such that $f(x) \leq cg(x)$ does not necessarily hold for all x (of course there must be some correction which is given by the algorithm). The algorithm is described in the attached paper, section 6.1. For this extra credit problem, you'll need to do two things:

a) Compute $E(M_t)$ for some large t $(= 1000)$ and small time step by simulating the whole path of Brownian motion.

b) Compute $E(M_t)$ using Metropolis-Hasting Acceptance-Rejection sampling.

c) Compare the two methods, which one is more efficient ? (Probably measured by the run time of the two respective methods).

d) If your answer is that a) is more efficient, do you think it is always more efficient no matter how large t is?