Homework 2 (Due 2/3/2017)

Math 622

February 6, 2017

1. Consider a market with 2 risky assets S^1, S^2 with dynamics under the physical measure $\mathbb{P}$:

$$\begin{aligned} dS_t^1 &= 2rS_t^1 dt + S_t^1(2W_t^1 + W_t^2) \\ dS_t^2 &= 0.5rS_t^2 dt + S_t^2(W_t^1 + 2W_t^2). \end{aligned}$$

Here W^1, W^2 are independent Brownian motions and r the risk free rate.

a) Is $\mathbb{P}$ a risk neutral probability?

Ans: No, since the drift term in S^1, S^2 are not r.

b) Is the model arbitrage free? Explain.

The market price of risk equation is

$$\begin{aligned} 2r - r &= 2\theta_1 + \theta_2 \\ 0.5 - r &= \theta_1 + 2\theta_2. \end{aligned}$$

This equation has a unique solution so the model is arbitrage free.

c) Is the model complete? Explain.

From part b, the model is complete.

2. (Correlation under change of measure)

Consider a two dimensional market model under the physical measure $\mathbb{P}$:

$$\begin{aligned} dS_t^1 &= \alpha^1 S_t^1 dt + S_t^1(\sigma_{11} dW_t^1 + \sigma_{12} dW_t^2) \\ dS_t^2 &= \alpha^2 S_t^2 dt + S_t^2(\sigma_{21} dW_t^1 + \sigma_{22} dW_t^2). \end{aligned}$$

Here W^1, W^2 are independent Brownan motions, all parametes are constants. Let the risk-free rate be r.

a) Assume there exists a risk neutral probability $\tilde{\mathbb{P}}$ and parameters θ_1, θ_2 such that

$$\widetilde{W^i}_t := W_t^i + \theta_i t, i = 1, 2$$

are Brownian motions under $\tilde{\mathbb{P}}$. Write down the equations that θ_1, θ_2 have to satisfy (the market price of risk equation).

Ans:

The market price of risk equation is

$$\alpha^i - r = \sigma_{i1}\theta_1 + \sigma_{i2}\theta_2, i = 1, 2.$$

b) Let

$$B_t^i = \frac{\sigma_{i1}W_t^1 + \sigma_{i2}W_t^2}{\sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}}, i = 1, 2.$$

Show that $B_t^i, i = 1, 2$ are Brownian motions.

Ans: B^i is a sum of martingales, so it is a martingale. Since W^1, W^2 are independent, their co-variation is 0. Thus the quadratic variation of B^i is just

$$\frac{\sigma_{i1}^2 t + \sigma_{i2}^2 t}{\sigma_{i1}^2 + \sigma_{i2}^2} = t.$$

Thus $B^i, i = 1, 2$ are Brownian motions by the Levy's characterization.

c) Denoting $\sigma_i = \sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}$ we can rewrite the above market model as:

$$\begin{aligned} dS_t^1 &= \alpha^1 S_t^1 dt + \sigma_1 S_t^1 dB_t^1 \\ dS_t^2 &= \alpha^2 S_t^2 dt + \sigma_2 S_t^2 dB_t^2. \end{aligned}$$

This effectively transforms the market model into one that only has one source of noise for each risky asset. However, note that B^1, B^2 here are NOT independent. Indeed, prove that

$$Cov(B_t^1, B_t^2) = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2} t \neq 0.$$

Since W^1, W^2 are independent,

$$\begin{aligned} Cov(B_t^1, B_t^2) &= Cov\left(\frac{\sigma_{11}W_t^1}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}}, \frac{\sigma_{21}W_t^1}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}}\right) + Cov\left(\frac{\sigma_{12}W_t^2}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}}, \frac{\sigma_{22}W_t^2}{\sqrt{\sigma_{11}^2 + \sigma_{12}^2}}\right) \\ &= \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2} t. \end{aligned}$$

d) Define

$$\gamma_i = \frac{\sigma_{i1}\theta_1 + \sigma_{i2}\theta_2}{\sigma_i},$$

where $\theta_i, i = 1, 2$ are defined in part a). Show that

$$\tilde{B}_t^i := B_t^i + \gamma_i t, i = 1, 2$$

are Brownian motions under $\tilde{\mathbb{P}}$.

Ans:

$$\begin{aligned}\tilde{B}_t^i &= B_t^i + \gamma_i t = \frac{\sigma_{i1}W_t^1 + \sigma_{i2}W_t^2}{\sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}} + \frac{\sigma_{i1}\theta_1 + \sigma_{i2}\theta_2}{\sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}}t \\ &= \frac{\sigma_{i1}\widetilde{W}_t^1 + \sigma_{i2}\widetilde{W}_t^2}{\sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}}.\end{aligned}$$

Now since $\widetilde{W}_t^i, i = 1, 2$ are Brownian motions under $\tilde{P}$ the argument for $\tilde{B}_t^i$ being Brownian motion is exctly the same as part b.

e) Show that the market model can be written as

$$\begin{aligned}dS_t^1 &= rS_t^1 dt + \sigma_1 S_t^1 d\tilde{B}_t^1 \\ dS_t^2 &= rS_t^2 dt + \sigma_2 S_t^2 d\tilde{B}_t^2.\end{aligned}$$

This is the risk neutral version of the model in part c).

We have

$$r + \sigma_i\gamma_i = r + \sigma_{i1}\theta_1 + \sigma_{i2}\theta_2 = r + \alpha^i - r = \alpha^i,$$

where the next to last equation is by part a. Thus this is the risk neutral version of the model in part c).

f) Show that the covariance of $\tilde{B}^1, \tilde{B}^2$ under $\tilde{\mathbb{P}}$ is the same as the covariance of B^1, B^2 under the physical measure B, which is

$$\frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}t.$$

Ans: $\tilde{B}_i$ and B_i has the same form, except we replace $\tilde{W}_i$ in place of W_i. Since $\tilde{W}_i$ (resp. W_i) are Brownian motions under P (resp $\tilde{P}$) it is obvious that they have the same covariance. The computation for the exact number is straightforward.

3. Shreve Exercise 5.14.

(i) Ans: Note that with dS_t as given

$$dX_t = \Delta_t(rS_t dt + \sigma S_t d\widetilde{W}_t) + r(X_t - \Delta_t S_t)dt.$$

Thus

$$\begin{aligned}d(e^{-rt}X_t) &= e^{-rt}(-rX_t dt + dX_t) \\ &= \Delta_t\sigma S_t d\widetilde{W}_t.\end{aligned}$$

Hence $e^{-rt}X_t$ is a $\tilde{P}$ martingale.

(ii) The equivlance of

$$Y_t = e^{\sigma \widetilde{W}_t + (r - \frac{1}{2}\sigma^2)t}$$

and

$$dY_t = rY_t dt + \sigma Y_t d\tilde{W}_t$$

and the implication of $e^{-rt}Y_t$ being a $\tilde{P}$ martingale is routine from Black-Scholes model.

To check the claim

$$S_t = S_0 Y_t + Y_t \int_0^t \frac{a}{Y_s} ds$$

satisfying (5.9.7) we see that

$$\begin{aligned}
dS_t &= S_0 dY_t + adt + (\int_0^t \frac{a}{Y_s} ds) dYt \\
&= adt + (S_0 + \int_0^t \frac{a}{Y_s} ds)(rY_t dt + \sigma Y_t d\tilde{W}_t) \\
&= adt + (S_0 Y_t + Y_t \int_0^t \frac{a}{Y_s} ds) rdt + (S_0 Y_t + Y_t \int_0^t \frac{a}{Y_s} ds) d\tilde{W}_t \\
&= adt + rS_t dt + \sigma S_t \tilde{W}_t
\end{aligned}$$

by definition of S_t.

(iii) Keeping in mind that $e^{-rt}Y_t$ is a $\tilde{P}$ martingale and for $s > t$

$$\frac{Y_T}{Y_s} = e^{\sigma(\widetilde{W}_T - \widetilde{W}_s) + (r - \frac{1}{2}\sigma^2)(T-s)}$$

is independent of $\mathcal{F}_t$ we have

$$\begin{aligned}
\tilde{E}(S_T|\mathcal{F}_t) &= S_0 \tilde{E}(Y_T|\mathcal{F}_t) + \tilde{E}(Y_T|\mathcal{F}_t) \int_0^t \frac{a}{Y_s} ds + a \int_t^T \tilde{E}(\frac{Y_T}{Y_s}|\mathcal{F}_t) ds \\
&= S_0 e^{r(T-t)} Y_t + e^{r(T-t)} Y_t \int_0^t \frac{a}{Y_s} ds + a \int_t^T e^{r(T-s)} ds \\
&= S_0 e^{r(T-t)} Y_t + e^{r(T-t)} Y_t \int_0^t \frac{a}{Y_s} ds + \frac{a(e^{r(T-t)} - 1)}{r} \\
&= e^{r(T-t)} S_t + \frac{a(e^{r(T-t)} - 1)}{r}.
\end{aligned}$$

(iv) Taking the differential of $e^{r(T-t)}S_t + \frac{a(e^{r(T-t)}-1)}{r}$ gives

$$e^{r(T-t)}(dS_t - rdt) - ae^{r(T-t)}dt \quad = \quad \sigma e^{r(T-t)} S_t d\tilde{W} t$$

is a martingale.

(v) From the result in part (iii)

$$\tilde{E}(e^{-r(T-t)}(S_T - K)|\mathcal{F}_t) \quad = \quad S_t + \frac{a(1-e^{-r(T-t)})}{r} - e^{-r(T-t)}K.$$

Thus

$$For_S(t,T) = e^{r(T-t)}S_t + \frac{a(e^{r(T-t)}-1)}{r} = Fut_S(t,T),$$

again from the result in part (iii).

(vi) We consider the dynamics of a portfolio π holding a long position in S ($\Delta_t = 1, 0 \leq \tau \leq T$) with $\pi_0 = 0$:

$$d\pi_t = dS_t - adt + (\pi_t - S_t)rdt.$$

We want to show $\pi_T = S_T - For_S(0,T) = S_T - e^{rT}S_0 + \frac{a(e^{rT}-1)}{r}$.

The dyanmics of π_t implies that

$$d(e^{-rt}\pi_t) = e^{-rt}\sigma S_t d\tilde{W}_t.$$

On the other hand,

$$d(e^{-rt}S_t) = e^{-rt}(-adt + \sigma S_t d\tilde{W}_t).$$

That is

$$d(e^{-rt}\pi_t) = d(e^{-rt}S_t) + e^{-rt}adt.$$

Integrating both sides gives:

$$e^{-rT}\pi_T = e^{-rT}S_T - S_0 + \frac{a(1-e^{-rT})}{r}$$

That is

$$\pi_T = S_T - e^{rT}S_0 + \frac{a(e^{rT}-1)}{r},$$

as required.

4. (An example of importance sampling) In this problem we want to compute $P(Z > 5)$ using the Monte Carlo simulation technique of importance sampling, at the heart of which is a change of measure. Here Z has standard Normal distribution. You will need to use some programming language that can simulate the normal distribution. You can choose whichever language you prefer.

a) Generate 10,000 independent standard Normal random random variables. Let n be the number of realizations that has value > 5. Report $n/10,000$.

Ans: Skip.

b) The answer you get in part a) should be 0. The reason is $\{Z > 5\}$ has such a small probability that even among 10,000 trials we are not likely to see 1 realization that has value above 5 (a 5-standard deviation event). Repeat the question in a) with 100,000 trials and report $n/100,000$.

Ans: Skip

c) The idea of importance sampling is to sample under a more suitable distribution. Suppose we want to calculate $E(h(X))$ where X has density function $f(x)$. Let $g(x)$ be another density function to be determined. Then

$$E(h(X)) = \int h(x)f(x)dx = \int \frac{h(x)f(x)}{g(x)} g(x)dx = E^g(\frac{h(X)f(X)}{g(X)}).$$

That is we instead compute the expectation of $\frac{h(X)f(X)}{g(X)}$ where X now has density $g(x)$ instead. Identify $h(x), f(x), g(x)$ in the example where we want to compute $P(Z > 5)$. To decide $g(x)$, let's say we want to sample instead under a $N(5, 1)$ distribution.

Ans:

$$\begin{aligned} h(x) &= \mathbf{1}_{\{x>5\}} \\ f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ g(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2}}. \end{aligned}$$

d) Use the formula in part c) to estimate $P(Z > 5)$ for 10,000 sample points. Report your answer.

Ans: Skip.

e) In class we discuss the change of measure kernel for a general Normal random variable X having $N(\mu, \sigma^2)$:

$$Z^\lambda = e^{\lambda X - \lambda\mu - \frac{1}{2}\lambda^2\sigma^2}$$

and

$$\tilde{\mathbb{E}}(Y) = \mathbb{E}(YZ^\lambda)$$

for any random variable Y. Show that the formula in part c) can be viewed according to this formulation as well. Identify Y, Z^λ when we want to estimate $P(Z > 5)$ (don't confuse Z and Z^λ here).

Ans: We showed that if X has $N(\mu, \sigma^2)$ distribution under P then X has $N(\mu + \lambda\sigma^2, \sigma^2)$ under $\tilde{P}$. We re-write the change of measure formula as

$$\mathbb{E}(Y) = \tilde{\mathbb{E}}(Y\frac{1}{Z^\lambda}).$$

Thus $Y = h(X) = \mathbf{1}_{\{X>5\}}$. We only need to verify that

$$Z^\lambda = e^{\lambda X - \lambda\mu - \frac{1}{2}\lambda^2\sigma^2} = \frac{f(X)}{g(X)},$$

where f, g are given in part c. We also have $\mu = 0, \sigma^2 = 1$ and $\mu + \lambda\sigma^2 = 5$ or $\lambda = 5$. Thus

$$Z^\lambda = e^{5X - \frac{25}{2}}.$$

On the other hand,

$$\frac{f(X)}{g(X)} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-5)^2}{2}}} = e^{5X - \frac{25}{2}}.$$

Thus the two formulations are equivalent.