Homework 1 (Due 1/25/2017)

Math 622

February 1, 2017

1. Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S_0^1 = 2, S_0^2 = 3$. There are 3 outcomes for S^1, S^2 at time T, denoted by $\omega_1, \omega_2, \omega_3$. Suppose that $S_T^1(\omega_1) = 3, S_T^1(\omega_2) = 1, S_T^1(\omega_3) = 2$ and $S_T^2(\omega_1) = 4, S_T^2(\omega_2) = 2, S_T^3(\omega_3) = 3$.

a. Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

Ans: A risk neutral probability exist, i.e. $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$. So the market is arbitrage free.

b. Is the market complete? Explain.

Ans: The market is not complete as the risk neutral probability is not unique. Indeed, another possible risk neutral probability would be $P(\omega_1) = P(\omega_2) = 1/6, P(\omega_3) = 2/3$.

2. Recall that $E(Y|\mathcal{F})$ is the unique $\mathcal{F}$-measurable random variable such that for any other $\mathcal{F}$-measurable random variable X:

$$E(YX) = E(E(Y|\mathcal{F})X). \tag{1}$$

Given a probability P, we define a new probability $\tilde{P}$ through a change of measure kernel Z (assuming Z satisfies all necessary conditions) by

$$\tilde{E}(Y) = E(YZ)$$

Show that

$$\tilde{E}(Y|\mathcal{F}) = \frac{E(YZ|\mathcal{F})}{E(Z|\mathcal{F})}. \tag{2}$$

This is the conditional change of measure formula. (Hint: Show that the RHS of (2) satisfies (1).)

Ans:

Let X be a $\mathcal{F}$-measurable random variable. Then

$$\tilde{E}(\frac{E(YZ|\mathcal{F})}{E(Z|\mathcal{F})}X) = E(\frac{E(YZ|\mathcal{F})}{E(Z|\mathcal{F})}XZ) = E(\frac{E(XYZ|\mathcal{F})}{E(Z|\mathcal{F})}Z).$$

Now

$$E(\frac{E(XYZ|\mathcal{F})}{E(Z|\mathcal{F})}Z) = E(E(\frac{E(XYZ|\mathcal{F})}{E(Z|\mathcal{F})}Z|\mathcal{F})) = E(E(XYZ|\mathcal{F})) = E(XYZ) = \tilde{E}(YX).$$

Thus for any $\mathcal{F}$ measurable random variable X

$$\tilde{E}(\frac{E(YZ|\mathcal{F})}{E(Z|\mathcal{F})}X) = \tilde{E}(YX).$$

By uniqueness of conditional expectation we have

$$\tilde{E}(Y|\mathcal{F}) = \frac{E(YZ|\mathcal{F})}{E(Z|\mathcal{F})}.$$

3. Let X be a Poisson (λ) variable. That is X takes values on the set of non-negative integers $(0, 1, 2, \cdots)$ and

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Define a random variable Z^u by

$$Z^u = e^{uX-\lambda(e^u-1)},$$

where u is a free parameter.

a) Show that $Z^u > 0$ and $E(Z^u) = 1$. That is Z^u is a proper change of measure kernel.

Ans: $Z^u > 0$ because exponential function is positive. $E(Z^u) = 1$ because of the moment generating function of a Poisson(λ) evaluated at u is $e^{\lambda(e^u-1)}$.

b) Define the probability $\tilde{P}$ by

$$\tilde{E}(Y) = E(YZ^u)$$

for any random variable Y. What is the distribution of X under $\tilde{P}$? (Hint: Compute the moment generating function of X under $\tilde{P}$).

Ans:

$$\begin{aligned}
\tilde{E}(e^{tX}) &= E(e^{tX}e^{uX-\lambda(e^u-1)}) = E(e^{(t+u)X})e^{-\lambda(e^u-1)} \\
&= e^{\lambda(e^{t+u}-1)-\lambda(e^u-1)} \\
&= e^{\lambda e^u(e^t-1)}.
\end{aligned}$$

Thus under $\tilde{P}$ X still has Poisson distribution with rate λe^u.

c) Is it possible to choose u so that under $\tilde{P}, X$ has Poisson(θ) distribution, where θ is an arbitrary positive parameter? If yes, show how. If not, explain.

Ans : It is possible by solving for $\lambda e^u = \theta$ and choose u accordingly. **4.** In this exercise we outline how to compute $E(W_t \int_0^t W_s ds)$.

a) Apply Ito's formula to find $d(W_t \int_0^t W_s ds)$.

Ans: Denoting $X_t = \int_0^t W_s ds$ then $d(W_t X_t) = X_t dW_t + W_t^2 dt$.

b) Use part a) to compute $E(W_t \int_0^t W_s ds)$.

$E(W_t \int_0^t W_s ds) = E(\int_0^t X_s dW_s + \int_0^t W_s^2 ds) = \int_0^t s ds = \frac{t^2}{2}$.

5. Let $Z_t = e^{-\theta W_t - \frac{1}{2}\theta^2 t}$ where W_t is a Brownian motion. This is the change of measure kernel we proposed for the Girsanov's theorem in class. Show that Z_t is a martingale.

We check directly

$$\begin{aligned} dZ_t &= -Z_t \frac{1}{2}\theta^2 dt - \theta Z_t dW_t + \frac{1}{2}\theta^2 Z_t dt \\ &= -\theta Z_t dW_t. \end{aligned}$$

Thus Z_t is a martingale.