Homework 9 (Due 4/7/2017)

Math 622

April 7, 2017

1. Let N be a Poisson process with rate λ and filtration $\mathcal{F}(t)$. Define

$$Y(t) := \exp\Big(uN(t) - \lambda t(e^u - 1)\Big),$$

that is Y is the exponential martingale associated with N. Use stochastic calculus (Ito's formula) for jump processes to show that

$$Y(t) = 1 + \int_0^t (e^u - 1)Y(s-)dM(s),$$

where $M(t) = N(t) - \lambda t$ and conclude that $Y(t)$ is a martingale w.r.t $\mathcal{F}(t)$.

Ans: Let $X(t) = uN(t) - \lambda t(e^u - 1)$. Then $Y(t) = e^{X(t)}$. By Ito's formula

$$Y(t) = 1 - \int_0^t \lambda Y(s)e^{u-1}ds + \sum_{s\le t} Y(s) - Y(s-).$$

Now

$$Y(s) - Y(s-) = Y(s-)(e^{u\Delta N(s)} - 1) = Y(s-)(e^u - 1)\Delta N(s)$$

and

$$\int_0^t \lambda Y(s)e^{u-1}ds = \int_0^t \lambda Y(s-)e^{u-1}ds.$$

These two facts together give the result.

2. Let $N_1(t), N_2(t)$ be independent Poisson processes with rate λ_1, λ_2 and $\mathcal{F}(t)$ a filtration for both N_1, N_2. Also define $M_i(t) = N_i(t) - \lambda_i t, i = 1, 2$.

(i) Use Ito's formula to show that

$$\begin{aligned} M_1(t)M_2(t) &= M_1(0)M_2(0) + \int_0^t M_1(s-)dM_2(s) + \int_0^t M_2(s-)dM_1(s) \\ &+ \sum_{s\le t}(M_1(s) - M_1(s-))(M_2(s) - M_2(s-)). \end{aligned}$$

Hint: Add and subtract $M_1(s)M_2(s-)$ to get

$$\begin{aligned}\sum_{s\le t} M_1(s)M_2(s) - M_1(s-)M_2(s-) &= \sum_{s\le t} M_1(s)(M_2(s) - M_2(s-)) \\ &+ \sum_{s\le t} M_2(s-)(M_1(s) - M_1(s-)).\end{aligned}$$

Ans: By Ito's formula and the hint

$$\begin{aligned}M_1(t)M_2(t) &= M_1(0)M_2(0) - \lambda_2 \int_0^t M_1(s)ds - \lambda_1 \int_0^t M_2(s)ds + \sum_{s\le t} M_1(s)(M_2(s) - M_2(s-)) \\ &+ \sum_{s\le t} M_2(s-)(M_1(s) - M_1(s-)) \\ &= M_1(0)M_2(0) - \lambda_2 \int_0^t M_1(s)ds - \lambda_1 \int_0^t M_2(s)ds + \sum_{s\le t} M_1(s-)(M_2(s) - M_2(s-)) \\ &+ \sum_{s\le t}(M_1(s) - M_1(s-))(M_2(s) - M_2(s-)) + \sum_{s\le t} M_2(s-)(M_1(s) - M_1(s-)) \\ &= M_1(0)M_2(0) - \lambda_2 \int_0^t M_1(s)ds - \lambda_1 \int_0^t M_2(s)ds + \int_0^t M_1(s-)dN_2(s) \\ &+ \int_0^t M_2(s-)dN_1(s) + \sum_{s\le t}(M_1(s) - M_1(s-))(M_2(s) - M_2(s-)) \\ &= M_1(0)M_2(0) + \int_0^t M_1(s-)dM_2(s) + \int_0^t M_2(s-)dM_1(s) \\ &+ \sum_{s\le t}(N_1(s) - N_1(s-))(N_2(s) - N_2(s-)).\end{aligned}$$

(ii) Conclude that the probability that N_1 and N_2 have the same jump time is 0 .

Take expectation on both sides, since both $\int_0^t M_i(s-)dM_j(s), i, j = 1, 2$ are martingales and M_1, M_2 are independent, we get

$$E(\sum_{s\le t}(N_1(s) - N_1(s-))(N_2(s) - N_2(s-))) = 0.$$

Since the expression inside the sum is non-negative, it means with probability 1, $(N_1(s) - N_1(s-))(N_2(s) - N_2(s-)) = 0$ for every s. This means either $N_1(s) - N_1(s-) = 0$ or $N_2(s) - N_2(s-) = 0$ for every s. That is N_1, N_2 cannot jump at the same time.

3. Let $N(t)$ be a Poisson(λ) process and W_t a Brownian motion. Let

$$\begin{aligned}X^u(t) &:= \exp\Big(uN(t) - \lambda t(e^u - 1)\Big), \\ Y^v(t) &:= \exp\Big(vW(t) - \frac{1}{2}v^2 t\Big).\end{aligned}$$

It is a fact that if $E(X^u(t)Y^v(t)) = E(X^u(t))E(Y^v(t))$ for any u, v then W_t and N_t are indepedendent. Show that this is the case by using Ito's formula on $X^u(t)Y^v(t)$, (utilize the fact that $X^u(t), Y^v(t)$ are martingales).

Ans: Using problem 1 and the usual Ito Calculus we know

$$X^u(t) = 1 + \int_0^t X^u(s-)(e^u - 1)d(N(s) - \lambda s)$$

and

$$dY^v(t) = vY^v(t)dW(t).$$

Thus by Ito's formula

$$\begin{aligned}
X^u(t)Y^v(t) &= 1 + \int_0^t vX^u(s)Y^v(s)dW(s) - \lambda(e^u - 1)\int_0^t Y^v(s)X^u(s-)ds \\
&+ \sum_{s\le t} X^u(s)Y^v(s) - X^u(s-)Y^v(s-) \\
&= 1 + \int_0^t vX^u(s)Y^v(s)dW(s) - \lambda(e^u - 1)\int_0^t Y^v(s)X^u(s-)ds \\
&+ \sum_{s\le t} Y^v(s)(X^u(s) - X^u(s-)) \\
&= 1 + \int_0^t vX^u(s)Y^v(s)dW(s) - \lambda(e^u - 1)\int_0^t Y^v(s)X^u(s-)ds \\
&+ \sum_{s\le t} Y^v(s)X^u(s-)(e^u - 1)\Delta N(s), \\
&= 1 + \int_0^t vX^u(s)Y^v(s)dW(s) - \int_0^t Y^v(s)X^u(s-)(e^u - 1)d(N(s) - \lambda ds).
\end{aligned}$$

Thus $E(X^u(t)Y^v(t)) = 1 = E(X^u(t))E(Y^v(t))$ for any u, v from which the conclusion follows.

Remark: What we actually have is

$$E(\exp(uN(t) + vW(t)) = \exp(\lambda t(e^u - 1))\exp(\frac{1}{2}v^2 t).$$

That is the joint moment generating functions equals the product of the individual moment generating function. This is the reason for independence. where the second equality comes from the fact that $Y(s)$ is continuous so $Y(s) = Y(s-)$.

4. Let $Y_1, Y_2, \cdots$ be i.i.d with distribution $P(Y_i = \frac{k}{10}) = 1/10, k = 1, 2, \cdots, 10$. Let $N(t)$ be a Poisson(1/2) process.

a) Simulate the compound Poisson process $Q(t) = \sum_{i=1}^{N(t)} Y_i$, for $t \in [0, 1]$.

b) Compute $E(Q(1))$ using simulation. Compare with the theoretical value.

5. Consider the following model for a stock with default :

$$dS_t = rS(t)dt + \sigma S(t)d\tilde{W}_t - S(t-)d(Q(t) - \mu dt), 0 \le t \le 1$$

where $Q(t) = \sum_{i=1}^{N(t)} Y_i$, Y_i is given in problem 4 but now $N(t)$ is a Poisson(1/10) process (so that the default happens only once over 10 years on average, a rare event) and $\mu = E(Q(1))$. To see that this indeed models default, note that when default happens (when $N(t)$ jumps) we have

$$S(t) - S(t-) = -S(t-)\Delta Q(t).$$

That is

$$S(t) = S(t-)(1 - \Delta Q(t)) = S(t-)(1 - Y_i).$$

Thus depending on the value of Y_i we can have the fall in stock value (percentage) of $10, 20 \cdots, 100\%$.

a) Let $S_0 = 50, \sigma = 0.1, r = 0.01$. Use simulation to compute $\tilde{E}(e^{-rT}(S_T - K)^+)$, the price of call option with $T = 1, K = 45$. (There is a theoretical expression for this, see lecture notes for more details).

b) Do you expect the Black-Scholes price of the same call option when S_t has no jump part to be higher or lower than the answer in part a) ? Compute this difference.

c) Repeat part a and b with European put option with strike $K = 55$ and expiry $T = 1$.

Discussion:

(i) This is a simplistic model of company default. One can easily adapt it to make it more realistic, i.e. change the distribution of Y_i (make it less likely to fall over a large percentage, for example), make the default rate dependent on the stock value (that is λ is a function of $S(t)$) etc.

(ii) The concept of credit spread is used to estimate the riskiness of a company. Usually it is measured as the difference in price of a similar instrument on a risky entity and a risk-free entity. In this sense, the difference in part b) and c) in problem 5 might be proposed as a measure for the credit spread. There are two problems with this:

First, the lack of benchmark. To be able to observe such a credit-spread in the market, we need to have a similar company (in terms of their volatility level) but is default-free, so that we can obtain their market-value call-option price (it is easy to see that there might be no such company). This is why corporate bond probably is a better instrument to measure credit risk since we have the benchmark of the government-issued bond, which is considered default-free.

Second, we perform the pricing under the risk-neutral framework. Thus the difference we come up with in part b) and c) are obtained under a risk-neutral point of view, and may not be proper to measure risk. Specifically, intuitively one might expect the call option price to be cheaper under defaultable model (since stock can only jump downward). In fact this may not be the case since the downward jump is compensated by the updrift μdt. A similar situation applies to the put option. In short, to measure risk one should work under the real-world probability (where no such compensation for $Q(t)$ occurs)).