Homework 8 (Due 3/31/2017)

Math 622

April 7, 2017

1. Let $\widetilde{P}$ be a risk neutral measure. Let S_t have the dynamics of geometric Brownian motion under $\widetilde{P}$:

$$dS_t = (0.5)S_t dt + \sqrt{2} S_t d\widetilde{W}_t.$$

Consider a perpetual American option on S_t with payoff function $g(S_t)$, where $g(x) = \left(K - \sqrt{x}\right)^+$ and K is a positive constant. Let V_t be the value of this American option at time t, assuming it has not been exercised.

a) By the Markov property of S_t, there is a bounded function $v(x)$ such that $v(S_t) = V_t$. State the equation for the obstacle problem that $v(x)$ satisfies. (You only need to state what it is and do not have to justify).

The obstacle problem is

$$\min(v(x) - (K - \sqrt{x})^+, (0.5)v - v_x(0.5)x - v_{xx}x^2) = 0.$$

b) Assuming that the continuation region has the form $C = \left\{x > L^*\right\}$ and exercise region has the form $E = \left\{0 \le x \le L^*\right\}$ where $L^* \le K^2$. By using the smooth pasting principle, solve for L^* and the solution $v(x)$ of the above problem. (Take care to verify that $v(x)$ is the actual solution.)

The general solution to

$$-(0.5)v + v_x(0.5)x + v_{xx}x^2 = 0$$

is

$$v(x) = Ax^{-0.5} + Bx.$$

Since $v(x)$ is bounded for $x > 0$, $v(x) = Ax^{-0.5}$. So the form of the solution is

$$\begin{aligned} v(x) &= K - \sqrt{x},\ 0 < x \le L^* \\ &= Ax^{-0.5},\ L^* < x. \end{aligned}$$

The smooth pasting conditions give

$$\begin{aligned} v(L^*) &= A(L^*)^{-0.5} = K - \sqrt{L^*} \\ v_x(L^*) &= -0.5A(L^*)^{-1.5} = -\frac{1}{2\sqrt{L^*}}. \end{aligned}$$

Solving for the above system gives

$$\begin{aligned} A &= L^* \\ L^* &= \left(\frac{K}{2}\right)^2. \end{aligned}$$

Thus

$$\begin{aligned} v(x) &= K - \sqrt{x},\ 0 < x \le \left(\frac{K}{2}\right)^2 \\ &= \frac{K^2}{4}x^{-.5},\ \left(\frac{K}{2}\right)^2 < x. \end{aligned}$$

Verification: We need to check that
(i) $\frac{K^2}{4}x^{-.5} \ge K - \sqrt{x}$ for $\left(\frac{K}{2}\right)^2 < x$
and
(ii) $v(x) = K - \sqrt{x}$ satisfies $(0.5)v - v_x(0.5)x - v_{xx}x^2 \ge 0$ for $0 < x \le \left(\frac{K}{2}\right)^2$.
For part (i) we apply the simple inequality $(a+b)^2 \ge 4ab$ to conclude

$$(\frac{K^2}{4}x^{-.5} + \sqrt{x})^2 \ge K^2,$$

which implies the desired inequality. As for part (ii) simply plugging in gives

$$(0.5)v - v_x(0.5)x - v_{xx}x^2 = 0.5(K - \sqrt{x}) + \frac{1}{2\sqrt{x}}(0.5)x - \frac{1}{4x^{3/2}}x^2 = 0.5(K - \sqrt{x}) \ge 0 \text{ for } x \le \left(\frac{K}{2}\right)^2.$$

2. Let

$$G(t) = \begin{cases} 2t & , \quad 0 \le t < 1; \\ t^2 - 3 & , \quad 1 \le t < 2; \\ t + 1 & , \quad 2 \le t. \end{cases}$$

Evaluate $\int_0^3 sdG(s)$.

Ans:

$$\begin{aligned}\int_0^3 sdG(s) &= \int_0^1 s(2ds) + 1(G(1) - G(1-)) + \int_1^2 s(2sds) + 2(G(2) - G(2-)) + \int_2^3 sds \\ &= 1 + (-2-2) + \frac{2}{3}(8-1) + 2(3-1) + \frac{5}{2} = \frac{49}{6}.\end{aligned}$$

3. Let N_t be a Poisson process. For this problem, it is important to remember that by definition, N_t is a right continuous with left limit process. For example, if we denote τ_1 as the time of the first jump of N_t then $N_{\tau_1 -} = 0$ and $N_{\tau_1} = 1$.

(i) Show that

$$\int_0^t N(u)dN(u) = \frac{N(t)(N(t)+1)}{2}.$$

Ans:

$$\int_0^t N(u)dN(u) = \sum_{s\le t} N(s)(N(s) - N(s-)) = \sum_{s\le t} N(s) = \sum_{k=1}^{N(t)} k = \frac{N(t)(N(t)+1)}{2}.$$

(ii) For a function $f(t)$, we denote $f(t-)$ to be the function that takes value as the left limit of $f(t)$ at every t. For example, if $f(t) = t$ then $f(t-) = t$ as well. On the other hand, the process $N(t-)$ will be a left continuous with right limit process. $N(t-) = N(t)$ if t is not a jump time of $N(t)$ and $N(t-) = N(t) - 1$ at the jump points of $N(t)$.

Show that

$$\int_0^t N(u-)dN(u) = \frac{N(t)(N(t)-1)}{2}$$

Ans:

$$\int_0^t N(u-)dN(u) = \sum_{s\le t} N(s-)(N(s) - N(s-)) = \sum_{s\le t} N(s-) = \sum_{k=0}^{N(t)-1} k = \frac{N(t)(N(t)-11)}{2}.$$

4. Simulate 100 sample points of an X random variable having $Poisson(2)$ distribution. Try both approaches using the inverse distribution function ($X = F^{-1}(U), U$ uniform [0,1]) and by viewing the $Poisson(2)$ random variable as the value of a Poisson(2) process at time $t = 1$ using exponential inter-arrival times (That is

$N_t = \{\max n : \sum_{i=1}^{n} \tau_i \leq t \}$ where τ_i are i.i.d exponential($\frac{1}{2}$)). Verify your simulation by checking that $E(X) = 2$ and $Var(X) = 2$ using the sample.

5. Simulate 100 paths of a Poisson (4) process $N_t, 0 \leq t \leq 1$. Try both approaches using the fact that the independent increments $dN(t)$ has distribution Poisson($4dt$) and using exponential inter-arrival times. Verify your result by computing $E(\int_0^1 tdN_t)$. It should be equal to $4\int_0^1 tdt$ (coming from the fact that $N_t - 4t$ is a martingale, which we'll discuss in the next lecture).

6. Extra credit (5 points for midterm 2, must be turned in before midterm 2) In class we discuss the analytic solution to the perpetual American put problem (that is we solve the obstacle problem directly). For the finite horizon problem in terms of computation we always have 2 options: the stochastic (simulation) approach and the PDE (numerical) approach. Can you find the stochastic approach for finding the solution to the perpetual put problem? (The challenge of course is $T = \infty$ so it's not clear what the stochastic approach should look like). Demonstrate by verifying that your approach agrees with the analytic solution we obtained before.