Math 622
Spring 2017
Final exam
5/9/17

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 2 pages of notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition of the interval $[0, T]$ such that $t_{i+1} - t_i = \delta, i = 0, \cdots, n-1$. Recall that the LIBOR rate, available for borrowing / lending during the interval $[t_i, t_{i+1}]$ is $L^\delta(t_i, t_i)$ (which is a random variable that is only known at or after time t_i). An investor wants to borrow 1 dollar with a fixed rate S (a constant) for the interval $[t_i, t_{i+1}]$. Thus he / she enters into a contract that pays $V^i_{t_{i+1}} = \delta(L^\delta(t_i, t_i) - S)$ to the holder at time t_{i+1}. With this contract, the investor can simply borrow 1 dollar at time t_i from the market at the variable rate $L(t_i, t_i)$. The net amount he / she "receives" at time t_{i+1} is exactly $-(1 + \delta S)$ which is the desired result.

   (a) (5 points) What is the value V^i_0 of this contract at time 0?

   Ans: Recall that the forward LIBOR rate $L^\delta(t, t_i)$ has the expression

$$L^\delta(t, t_i) = \frac{1}{\delta}\left(\frac{B(t, t_i)}{B(t, t_{i+1})} - 1\right).$$

   Under the forward measure $\tilde{P}^{t_{i+1}}$, $\frac{B(t,t_i)}{B(t,t_{i+1})}$ is a martingale.

   Thus using the forward measure $\tilde{P}^{t_{i+1}}$ we have

$$\begin{aligned} V^i_0 &= \tilde{E}^{t_{i+1}}[\delta(L(t_i, t_i) - S)]B(0, t_{i+1}) = \tilde{E}^{t_{i+1}}[\frac{B(t_i, t_i)}{B(t_i, t_{i+1})} - 1 - \delta S)]B(0, t_{i+1}) \\ &= [\frac{B(0, t_i)}{B(0, t_{i+1})} - 1 - \delta S)]B(0, t_{i+1}) \\ &= B(0, t_i) - B(0, t_{i+1}) - \delta S B(0, t_{i+1}). \end{aligned}$$

   Alternativel, recall that the forward LIBOR rate $L^\delta(0, t_i)$ is a free to enter contract (quoted at time $t = 0$) that allows the investor to lend / borrow on the interval $[t_i, t_{i+1}]$ with rate $L^\delta(0, t_i)$. The cost to borrow at a fixed rate S different from $L^\delta(0, t_i)$ is exactly the difference in the amount earned between these two rates discounted back to time 0. That is

$$V^i_0 = B(0, t_{i+1})(L^\delta(0, t_i) - S)\delta$$

   which gives the same answer as above.

   (b) (5 points) The investor wants to borrow 1 dollar at each time t_i for the period $[t_i, t_{i+1}]$ at a fixed rate S. So he / she simply buys a contract that provides a cashflow at each time t_{i+1} equalling $\delta(L(t_i, t_i) - S)$ for $i = 0, 1, 2, \cdots, n-1$ (This is also known as an interest rate swap). What is the value of this contract V_0 at time 0?

   Ans: The value V_0 is simply the sum of $V^i_0, i = 0, 1, \cdots, n-1$ namely

$$\begin{aligned} V_0 &= \sum_{i=0}^{n-1}[B(0, t_i) - B(0, t_{i+1}) - \delta S B(0, t_{i+1})] \\ &= 1 - B(0, T) - \delta S \sum_{i=0}^{n-1} B(0, t_{i+1}). \end{aligned}$$

   (c) (10 points) There is a fixed rate S_0, called the swap rate, such that the interest rate swap is free to enter at time 0. Find S_0.

Using part b, we see that

$$S_0 = \frac{1 - B(0,T)}{\delta \sum_{i=0}^{n-1} B(0, t_{i+1})}.$$

2. Consider the following Hull-White model for interest rate under the risk neutral measure:

$$\begin{aligned} dR_t &= -R_t dt + d\tilde{W}_t \\ R_0 &= 0. \end{aligned}$$

(a) (5 points) Find the explicit solution for R_t.
Ans:

$$d(e^t R_t) = e^t d\tilde{W}_t.$$

Thus

$$R_t = \int_0^t e^{u-t} d\tilde{W}_u.$$

(b) (5 points) Express $\int_t^T R_s ds$ as an expression involving Ito integral (switching the order of ds, dW_u is needed). What is the distribution of $\int_t^T R_s ds$?

$$\begin{aligned} \int_t^T R_s ds &= \int_t^T (R_t e^{-s} + \int_t^s e^{u-s} d\tilde{W}_u) ds \\ &= R_t \int_t^T e^{-s} ds + \int_t^T (\int_t^s e^{u-s} d\tilde{W}_u) ds \\ &= R_t \int_t^T e^{-s} ds + \int_t^T (\int_u^T e^{u-s} ds) d\tilde{W}_u. \end{aligned}$$

Thus $\int_t^T R_s ds$ has $N(\mu_t R_t, \sigma_t^2)$ distribution where

$$\begin{aligned} \mu_t &= \int_t^T e^{-s} ds \\ \sigma_t^2 &= \int_t^T (\int_u^T e^{u-s} ds)^2 du) \end{aligned}$$

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(c) (5 points) This is an affine yield model so that

$$B(t,T) = e^{-A(t,T) - C(t,T) R_t}.$$

Find $A(t,T), C(t,T)$ (use the answer in the previous part and the moment generating function of a Normal RV).
Ans:

$$\begin{aligned} B(t,T) &= \tilde{E}(e^{-\int_t^T R_s ds} | \mathcal{F}_t) \\ &= e^{-\mu_t R_t - \frac{1}{2}\sigma_t^2}. \end{aligned}$$

Thus $C(t,T) = \mu_t$ and $A(t,T) = \frac{1}{2}\sigma_t^2$.

(d) (5 points) Find $df(t,T)$ the dynamics of the forward rate under the risk neutral measure. Suppose

$$df(t,T) = \alpha(t,T)dt + \gamma(t,T)d\tilde{W}_t.$$

We have

$$\int_t^T f(t,u)du = \mu_t R_t + \frac{1}{2}\sigma_t^2.$$

On the other hand

$$\begin{aligned} d\int_t^T f(t,u)du &= -R(t)dt + \int_t^T df(t,u)du \\ &= (\alpha^*(t,T) - R(t))dt + \gamma^*(t,T)d\tilde{W}_t. \end{aligned}$$

And

$$\begin{aligned} d(\mu_t R_t + \frac{1}{2}\sigma_t^2) &= \dot{\mu}_t R_t dt + \mu_t dR_t + \sigma_t \dot{\sigma}_t dt \\ &= [(\dot{\mu}_t - \mu_t)R_t + \sigma_t\dot{\sigma}_t]dt + \mu_t d\tilde{W}_t. \end{aligned}$$

Where $\dot{\mu}_t$ and $\dot{\sigma}_t$ refer to derivatives with respect to t. Equating the two dynamics we see that

$$\begin{aligned} \gamma^*(t,T) &= \mu_t \\ \alpha^*(t,T) &= R(t) + (\dot{\mu}_t - \mu_t)R_t + \sigma_t\dot{\sigma}_t. \end{aligned}$$

3. (20 points) Consider the following model for the Euro-dollar exchange rate under the physical measure:

$$dQ_t = \mu Q_t dt + \sigma Q_t dW_t.$$

That is Q_t is the price (in dollar) of 1 Euro at time t. Consider a call option on 1 dollar quoted in Euro :

$$V_T = (\frac{1}{Q_T} - K)^+,$$

where K is, of course, in Euro. Let the US risk free rate be r and the Euro risk free rate be r^f. Find the explicit expression for V_0 (either in dollar or in Euro) using the Black-Scholes formula approach.

Ans: Letting $Q_t^f := \frac{1}{Q_t}$ We have

$$dQ_t^f = (r^f - r)Q_t dt - \sigma Q_t^f d\tilde{W^f}_t.$$

Thus

$$Q_T^f = Q_0^f e^{(r^f - r - \frac{1}{2}\sigma^2)T - \sigma \tilde{W^f}_T}.$$

$$\begin{aligned} V_0 &= \tilde{E}^f[e^{-r^f T}(Q^f - K)^+] \\ &= e^{-rT}\tilde{E}^f[e^{-(r^f - r)T}(Q^f - K)^+] \\ &= e^{-rT}[Q_0^f N(d+) - Ke^{-(r^f - r)T}N(d-)] \end{aligned}$$

where

$$N(d\pm) = \frac{(r^f - r \pm \frac{1}{2}\sigma^2)T - \log(\frac{K}{Q_0^f})}{\sigma\sqrt{T}}.$$

4. (20 points) Consider the equation

$$dX_t = X_t dt + dN_t,$$

where N_t is a Poisson (λ) process. Find $E(X_1)$.

Ans: We have

$$dX_t = X_t dt + \lambda dt + d(N_t - \lambda t).$$

Thus

$$X_t = e^t X_0 + \lambda \int_0^t e^{t-u} du + \int_0^t e^{-u} d(N_u - \lambda u).$$

Hence

$$E(X_1) = eX_0 + \lambda \int_0^1 e^{1-u} du.$$

5. (20 points) Consider a geometric Poisson model for the stock under a risk neutral measure:

$$dS_t = rS_t dt + \sigma S_{t-} d(N_t - \lambda t),$$

where N_t is a Poisson (λ) process. Let $V_T = \int_0^T S_u du$. Find an explicit formula for V_0.

Ans:

$$\begin{aligned} V_0 &= \tilde{E}(e^{-rT} \int_0^T S_u du) \\ &= e^{-rT} \int_0^T \tilde{E}(S_u du) \\ &= e^{-rT} \int_0^T e^{ru} \tilde{E}(e^{-ru} S_u du) \\ &= e^{-rT} \int_0^T S_0 e^{ru} \\ &= S_0 \frac{1 - e^{-rT}}{r}, \end{aligned}$$

where for the fourth equality we have used the fact that $e^{-rt} S_t$ is a martingale.

Scratch (Won't be graded)

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