Math 421
Spring 2016
Midterm exam 2
4/14/17

Name (Print): ____________________

This exam contains 9 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total:	100	

1. Consider V the set of functions defined on $[-\pi, \pi]$ such that their second derivative is 1 : $f''(x) = 1, f$ defined on $[-\pi, \pi]$.

   (a) (5 points) Give an example of such a function in V.
   Ans: $f(x) = \frac{1}{2}x^2$.

   (b) (10 points) Is V a vector space? Explain.
   V is not a vector space. The zero function is not in it. It is not closed under scalar multiplication and addition : if $g(x) = 2f(x)$ and $f(x)$ is in V then $g''(x) = 2$ so g cannot be in V.

2. (15 points) Use Laplace transform to solve the following equation:

$$y' + y = \delta(t-1), y(0) = 2.$$

Sol:

$$sY(s) - 2 + Y(s) = e^{-s}.$$

That is

$$Y(s) = \frac{e^{-s} + 2}{s+1}.$$

Taking the inverse transform gives

$$y(t) = e^{-t-1}\mathcal{U}(t-1) + 2e^{-t}$$

3. (15 points) Let $f(x)$ defined on $-\pi < x < \pi$ as

$$\begin{aligned} f(x) &= 1, -\pi < x < 0 \\ &= 1-x, 0 \leq x < \pi. \end{aligned}$$

Find the Fourier series representation of $f(x)$.

Sol: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}(\pi + \int_0^{\pi}(1-x)dx) \\ &= \frac{1}{\pi}(2\pi - \frac{\pi^2}{2}). \end{aligned}$$

And

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ &= \frac{-1}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{-1}{\pi}(x\frac{sin(nx)}{n}|_0^{\pi} - \int_0^{\pi}\frac{sin(nx)}{n}dx) \\ &= \frac{-1}{\pi n^2}\cos(nx)|_0^{\pi} \\ &= \frac{1}{\pi n^2}(1-\cos(n\pi)). \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\ &= \frac{-1}{\pi}\int_0^{\pi} x\sin(nx)dx \\ &= \frac{-1}{\pi}(-\frac{x\cos nx}{n}|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx) \\ &= \frac{\cos(n\pi)}{n} \end{aligned}$$

4. (15 points) Let $f(x) = \pi - x$ defined on $0 < x < \pi$. Find the Fourier half-range sine expansion for $f(x)$.

   Ans:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

   where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)dx \\ &= \frac{2}{\pi}(-\pi\frac{\cos(nx)}{n}|_0^{\pi} - \int_0^{\pi} x\sin(nx)dx) \\ &= \frac{2}{\pi}(\frac{\pi}{n}(1-\cos(n\pi)) + \frac{\pi\cos(n\pi)}{n}) \\ &= \frac{2}{n}. \end{aligned}$$

   where the computation for $\int_0^{\pi} x\sin(nx)$ was done in problem 3.

5. (20 points) Find a particular solution to the equation:

$$y'' + y = f(t),$$

where $f(t) = \pi - t, 0 < t < 2\pi$ and $f(t)$ is periodic with period 2π. Does resonance happen in this problem?

Using the sine expansion in the previous problem we have

$$f(t) = \sum_{n=1} \infty \frac{2}{n} \sin(nt).$$

Thus assuming that $y_p(t) = \sum_{n=1} \infty B_n \sin(nt)$ we have

$$\sum_{n=1}^{\infty} (1 - n^2) B_n \sin(nt) = \frac{2}{n} \sin(nt).$$

This has solution $B_n = \frac{2}{n(1-n^2)}$ for $n \neq 1$. When $n = 1$ we assume $y_1(t) = At \cos t + Bt \sin t$. Plugging this in to solve for

$$y_1'' + y_1 = \frac{2}{n} \sin(t)$$

we have $A = \frac{2}{n}, B = 0$. Thus a particular solution is

$$y_p(t) = \sum_{n=2}^{\infty} \frac{2}{n(1 - n^2)} \sin(nt) + \frac{2}{n} tsint.$$

6. (20 points) Consider the Sturm-Liouville problem on the interval $[0, L]$

$$\begin{aligned} y'' + \lambda y &= 0, 0 < x < L \\ y(0) &= y'(L) = 0. \end{aligned}$$

Note : it's the Dirichlet condition on the left hand side and Neumann condition on the right hand side. Find the eigenvalues and eigenfunctions of this problem.

Case : $\lambda = 0$

$$y(x) = C_1 + C_2 x.$$

$y(0) = 0$ implies $C_1 = 0$. $y'(L) = 0$ implies $C_2 = 0$. Thus $\lambda = 0$ is not an eigenvalue.

Case : $\lambda = -\alpha^2 < 0$

$$y(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}.$$

$$\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y'(L) &= \alpha C_1 e^{\alpha L} - \alpha C_2 e^{-\alpha L} = 0. \end{aligned}$$

The determinant of the systme is $-\alpha e^{-\alpha L} - \alpha e^{\alpha L}$ is 0 if and only if $\alpha = 0$. Thus there is no eigenvalue for the negative λ case.

Case: $\lambda = \alpha^2 > 0$

$$y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x).$$

$y(0) = 0$ implies $C_1 = 0$. $y'(L) = 0$ implies $\alpha C_2 \cos(\alpha L) = 0$. $\cos(\alpha L) = 0$ when $\alpha L = \frac{n\pi}{2}, n = 1, 3, 5, \cdots$. Thus the eigenvalues for the problem is $\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1, 3, 5, \cdots$ with corresponding eigenfunctions $v_n = \sin(\frac{n\pi}{2}x)$.

Scratch (Won't be graded)

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