Math 421 **Name (Print):** ______________________
Spring 2016
Midterm exam 1
3/3/17

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	20	
7	20	
Total:	100	

1. (10 points) Solve the linear system

$$\begin{aligned} 2x_1 + 3x_2 + -2x_3 &= -7 \\ 4x_1 + x_2 + 3x_3 &= 5 \\ 2x_1 - 5x_2 + 7x_3 &= 19. \end{aligned}$$

Ans:

$$\begin{bmatrix} 2 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & -2 & -7 \\ 0 & -5 & 7 & 19 \\ 2 & -5 & 7 & 19 \end{bmatrix}.$$

The last two rows imply $x_1 = 0$. The system reduces to:

$$\begin{aligned} 3x_2 + -2x_3 &= -7 \\ x_2 + 3x_3 &= 5. \end{aligned}$$

$-3R2 + R1$ gives $-11x_3 = -22$ or $x_3 = 2$. This gives $x_2 = -1$.

2. (10 points) Let

$$A^{-1} \quad = \quad \begin{bmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Find A.

Ans: Some steps of row reductions are shown below:

$$\begin{bmatrix} -1 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 5 & 6 & -3 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

3. (10 points) Let

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}.$$

Find all eigenvalues and corresponding eigenvectors of A.

Ans: The eigenvalues satisfy $(4-\lambda)(1-\lambda)-10=0$. That is $\lambda^2-5\lambda-6=0$. Thus $\lambda = 6, -1$.

If $\lambda = 6$, $A - \lambda I = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If $\lambda = -1$, $A - \lambda I = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$. Thus an eigenvector is $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

4. (15 points) Solve the initial value problem

$$y' + 6y = e^{4t}, y(0) = 2.$$

Ans: The Laplace transform is

$$sY(s) - 2 + 6Y(s) = \frac{1}{s-4}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1}{(s-4)(s+6)} + \frac{2}{s+6} \\ &= \frac{1/10}{s-4} + \frac{19/10}{s+6}. \end{aligned}$$

That is $y(t) = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}$.

5. (15 points) Solve the initial value problem

$$y' + y = f(t), y(0) = 1$$

where

$$\begin{aligned} f(t) &= 0, 0 \leq t < \pi \\ &= \cos t, t \geq \pi. \end{aligned}$$

Ans: $y' + y = \mathcal{U}(t-\pi)\cos(t) = y' + y = -\mathcal{U}(t-\pi)\cos(t-\pi)$.
Taking Laplace transform gives

$$sY(s) - 1 + Y(s) = \frac{-s}{s^2+1}e^{-\pi s}.$$

Thus

$$Y(s) = \frac{-s}{(s^2+1)(s+1)}e^{-\pi s} + \frac{1}{1+s}.$$

We have

$$\frac{s}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}.$$

We have $A(s^2+1) + (Bs+C)(s+1) = s$. Plus in $s = -1$ gives $A = \frac{-1}{2}$. Plug in $s = 0$ gives $C = \frac{1}{2}$. Finally $B = -A = \frac{1}{2}$.
Thus $y(t) = -[Ae^{-(t-\pi)} + B\cos(t-\pi) + C\sin(t-\pi)]\mathcal{U}(t-\pi) + e^{-t}$.

6. (20 points) Solve the initial value problem

$$y' - y = e^t \sin t, y(0) = 0.$$

Ans:

$$sY(s) - Y(s) = \frac{1}{(s-1)^2+1}.$$

Thus

$$Y(s) = \frac{1}{((s-1)^2+1)(s-1)} = \frac{As+B}{(s-1)^2+1} + \frac{C}{s-1}.$$

We have $(As+B)(s-1) + C((s-1)^2+1) = 1$. Plug in $s = 1$ gives $C = 1$. Plug in $s = 0$ gives $B = 1$. Finally $A = -C = -1$. Thus

$$Y(s) = -\frac{s-1}{(s-1)^2+1} + \frac{1}{s-1}.$$

Thus

$$y(t) = -e^t \cos t + e^t.$$

7. (20 points) Let
$$f(t) = \frac{s+1}{s^3(s+1)^2}.$$
Find the partial fraction decomposition of $f(t)$.

Ans:
$$f(t) = \frac{1}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}.$$

We have $D = \frac{1}{s^3}|_{s=-1} = -1$. $C = \frac{1}{s+1}|_{s=0} = 1$. $B = \lim_{s\to 0} s^2(\frac{1}{s^3(s+1)} - \frac{1}{s^3}) = -1$. Finally plug in $s = 1$ gives $A = 1$.

Scratch (Won't be graded)

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