Math 421
Spring 2016
Final exam
5/9/17

Name (Print): ______________________________

This exam contains 9 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (two sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Find the solution to the system

$$\begin{aligned} x + 3y - 2z &= -7 \\ 4x + y + 3z &= 5 \\ 2x - 5y + 7z &= 19. \end{aligned}$$

Ans:

$$\begin{bmatrix} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{bmatrix}$$

reduces to

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus letting z be the free variable, the general solution is $[2 - z, z - 3, z]$.

2. (10 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{bmatrix},$$

where a, b, c are some constants. (There are easier ways to answer this question than expanding using the determinant rule. Expansion still works if you keep your arithmetic right).

Ans: Replacing R3 with the sum of R2 and R3 does not change the determinant. On the other hand,

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{bmatrix}$$

has determinant $det(A') = 0$ because R1 and R3 are dependent. Thus the original determinant is also 0.

3. (10 points) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 4 & 8 \\ 0 & -5 \end{bmatrix}.$$

Ans: The equation $(4 - \lambda)(-5 - \lambda) = 0$ has solutions $\lambda = 4, -5$. These are the eigenvalues. If $\lambda = 4$, $A - \lambda I$ is

$$\begin{bmatrix} 0 & 8 \\ 0 & -9 \end{bmatrix}.$$

Thus an eigenvector is $[1,0]^T$. If $\lambda = -5, A - \lambda I$ is

$$\begin{bmatrix} 9 & 8 \\ 0 & 0 \end{bmatrix}.$$

Thus an eigenvector is $[-8,9]^T$.

4. (10 points) Solve the following initial value problem for $y(t)$:

$$\begin{aligned} y'' + 4y &= \cos(2t), \\ y(0) &= 1, y'(0) = 0. \end{aligned}$$

Taking Laplace transform gives

$$s^2Y(s) - s + 4Y(s) = \frac{s}{s^2+4}.$$

Thus

$$Y(s) = \frac{s}{s^2+4} + \frac{s}{(s^2+4)^2}.$$

The derivative of $\frac{2}{s^2+4}$ is $-\frac{4s}{(s^2+4)^2}$. Thus

$$y(t) = \cos(2t) + \frac{1}{4}t\sin(2t).$$

5. (10 points) Solving the following Volterra integral equation for $f(t)$:

$$f(t) + \int_0^t (t-s)f(s)ds = 1.$$

Ans: Taking Laplace transform gives

$$F(s) + \frac{F(s)}{s^2} = \frac{1}{s}.$$

Thus

$$F(s) = \frac{s}{s^2+1}.$$

That is $f(t) = \cos(t)$.

6. (10 points) Find a particular solution to the equation:

$$y'' + 4y = f(t),$$

where $f(t) = \begin{cases} 1 & , \quad 0 < x < \pi/2 \\ -1 & , \quad \pi/2 < x < \pi \end{cases}$ and $f(t)$ is periodic with period 2π. Does resonance happen in this problem?

Ans: The sine half-range expansion of $f(t)$ is $f(t) = \sum_{n=1}^{\infty} B_n \sin(nt)$ where

$$\begin{aligned} B_n &= \frac{2}{\pi}\left(\int_0^{\pi/2} \sin(nt)dt - \int_{\pi/2}^{\pi} \sin(nt)dt\right) \\ &= \frac{2}{n\pi}[\sin(n\frac{\pi}{2}) - (\sin(n\pi) - \sin(n\frac{\pi}{2})] \\ &= \frac{4\sin(n\frac{\pi}{2})}{n\pi}. \end{aligned}$$

Let $y(t) = \sum_{n=1}^{\infty} y_n(t)$, where $y_n(t) = A_n \sin(nt), n \neq 2$ and plug in we have

$$A_n(4 - n^2) = B_n, n \neq 2.$$

Thus

$$A_n = \frac{B_n}{4 - n^2}, n \neq 2.$$

For $n = 2$, we have

$$y_2(t) = -\frac{B_2}{4} t \cos t.$$

Resonance happens in this problem.

7. Consider the set V of continuous functions on defined on $[0, 1]$ with the property that

$$\int_0^1 f(x)dx = 0$$

for any f in V.

(a) (5 points) Give an example of a member of V (note that the function must be continuous).
Ans: $f(x) = \frac{1}{2} - x$.

(b) (5 points) Is V a vector space? Explain.
Ans: V is a vector space: The function $f(x) = 0$ is in V, it is close under addition and constant multiplication.

8. (10 points) Solve the following heat equation

$$u_t = u_{xx}, t > 0, 0 < x < \pi$$

with initial condition

$$u(0, x) = \begin{cases} 1 & , \quad 0 < x < \pi/2 \\ 0 & , \quad \pi/2 < x < \pi \end{cases}$$

and non-homogeneous Dirichlet condition

$$u(t,0) = 0, u(t,\pi) = 1.$$

(Explicit Fourier coefficient is required).

Ans: Let $u(t,x) = v(t,x) + \phi(x)$ where $v(t,x)$ solves

$$\begin{aligned} v_t &= v_{xx}, t > 0, 0 < x < \pi \\ v(0,x) &= u(0,x) - \phi(x) \\ v(t,0) &= v(t,\pi) = 0. \end{aligned}$$

Then we see that $\phi(x)$ satisfies

$$\phi''(x) = 0, \phi(0) = 0, \phi(\pi) = 1.$$

Thus $\phi(x) = \frac{x}{\pi}$. Now

$$v(t,x) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx),$$

where

$$B_n = \frac{2}{\pi} \int_0^{\pi} (u(0,x) - \frac{x}{\pi}) \sin(nx) dx.$$

We have

$$\int_0^{\pi} (u(0,x) \sin(nx) dx = \int_0^{\pi/2} \sin(nx) dx = \frac{1 - \cos(\frac{n\pi}{2})}{n}.$$

On the other hand,

$$\int_0^{\pi} x \sin(nx) dx = -\frac{\cos(nx)}{n} x\Big|_0^{\pi} + \int_0^{\pi} \cos(nx) dx = -\frac{\pi \cos(n\pi)}{n}.$$

9. (10 points) Solve the following non-homogeneous wave equation

$$u_{tt} = u_{xx} - g, t > 0, 0 < x < \pi$$

were g is the gravitation constant with initial conditions

$$\begin{aligned} u(0,x) &= \begin{cases} 1 & , \quad 0 < x < \pi/2 \\ 0 & , \quad \pi/2 < x < \pi \end{cases} \\ u_t(0,x) &= 0. \end{aligned}$$

and homogenous Dirichlet condition

$$u(t,0) = u(t,\pi) = 0.$$

(Explicit Fourier coefficient is required).

Ans: Let $u(t,x) = v(t,x) + \phi(x)$ where $v(t,x)$ solves

$$\begin{aligned} v_{tt} &= v_{xx}, t > 0, 0 < x < \pi \\ v(0,x) &= u(0,x) - \phi(x) \\ v_t(0,x) &= 0 \\ v(t,0) &= v(t,\pi) = 0. \end{aligned}$$

Then we see that $\phi(x)$ satisfies

$$\phi''(x) = g, \phi(0) = 0, \phi(\pi) = 0.$$

Thus $\phi(x) = \frac{g}{2}x(\pi - x)$. Now

$$v(t,x) = \sum_{n=1}^{\infty} A_n \cos(nt)\sin(nx) + \sum_{n=1}^{\infty} B_n \sin(nt)\sin(nx),$$

where $B_n = 0$ because $v_t(0,x) = 0$. Thus

$$A_n = \frac{2}{\pi}\int_0^{\pi} (u(0,x) - \frac{g}{2}x(\pi - x))\sin(nx)dx.$$

We computed $\int_0^{\pi} u(0,x)\sin(nx)dx$ and $\int_0^{\pi} x\sin(nx)dx$ in the previous problem. It remains to compute

$$\begin{aligned} \int_0^{\pi} x^2 \sin(nx)dx &= -\frac{x^2}{n}\cos(nx)\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos(nx)dx \\ &= -\frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n}\int_0^{\pi} x\cos(nx)dx. \end{aligned}$$

And

$$\begin{aligned} \int_0^{\pi} x\cos(nx)dx &= \frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx \\ &= \frac{1}{n^2}(1 - \cos(n\pi)). \end{aligned}$$

10. (10 points) Solve the following Laplace equation

$$u_{xx} + u_{yy} = 0, 0 < x < \pi, 0 < y < \pi$$

with non-homogeneous Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= u(\pi,y) = 0 \\ u(x,\pi) &= u(0,y) = 1. \end{aligned}$$

(Explicit Fourier coefficient is required).

Ans: $u(x,y) = u^1(x,y) + u^2(x,y)$ where $u_1(x,y)$ solves

$$\begin{aligned} u^1_{xx} + u^1_{yy} &= 0, 0 < x < \pi, 0 < y < \pi \\ u^1(x,0) = u^1(0,y) &= 0 \\ u^1(x,\pi) = 0, u^1(\pi,y) &= 1 \end{aligned}$$

and $u_2(x,y)$ solves

$$\begin{aligned} u^2_{xx} + u^2_{yy} &= 0, 0 < x < \pi, 0 < y < \pi \\ u^2(x,0) = u^2(0,y) &= 0 \\ u^2(x,\pi) = 1, u^2(\pi,y) &= 0. \end{aligned}$$

We have

$$u^2(x,y) = \sum_{n=1}^{\infty} A_n \sinh(ny)\sin(nx)$$

where

$$A_n = \frac{2}{\sinh(n\pi)} \int_0^{\pi} \sin(nx)dx = \frac{2(1-(-1)^n)}{\sinh(n\pi)}.$$

By symmetry between x, y we have

$$u^1(x,y) = \sum_{n=1}^{\infty} B_n \sinh(nx)\sin(ny)$$

where

$$B_n = \frac{2}{\sinh(n\pi)} \int_0^{\pi} \sin(ny)dy = \frac{2(1-(-1)^n)}{\sinh(n\pi)}.$$

Scratch (Won't be graded)

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