Math 622
Spring 2016
Midterm exam 2
4/13/16

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 2 pages of notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Black-Scholes formula:

$$E\Big[e^{-rt}(xe^{(r-\frac{1}{2}\sigma^2 t)+\sigma W_t} - K)^+\Big] = xN(d+) - Ke^{-rt}N(d-),$$

where

$$d\pm = \frac{(r \pm \frac{1}{2}\sigma^2)t - \log(K/x)}{\sigma\sqrt{t}}.$$

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (a) (5 points) Let $W(t)$ be a Brownian motion and $W(0) = 0$. Let $b_1 > 0$ and $b_2 < 0$ be 2 constants. Define

$$\tau_1 := \inf\{t \geq 0 : W(t) = b_1 \text{ or } W(t) = b_2\}.$$

Is τ_1 a stopping time with respect to $\mathcal{F}^W(t)$? Provide a brief explanation for your answer.
Ans: Denote

$$\widetilde{\tau}_1 := \inf\{t : W(t) = b_1\}$$
$$\bar{\tau}_1 := \inf\{t : W(t) = b_2\}$$

Then $\widetilde{\tau}_1$ and $\bar{\tau}_1$ are stopping times, $\tau_1 = \widetilde{\tau}_1 \wedge \bar{\tau}_1$ so τ_1 is a stopping time.

(b) (5 points) Let

$$\begin{aligned} S_t &= rS_t dt + S_t dW_t; \\ S(0) &= 1. \end{aligned}$$

Define

$$\tau_2 := \sup\{t \geq 0 : \int_0^t S(u)du = 2\}.$$

Is τ_2 a stopping time with respect to $\mathcal{F}^S(t)$? Provide a brief explanation for your answer.
Ans:
Note that $S(u) > 0, \forall u$ therefore if we denote

$$Y_t := \int_0^t S(u)du$$

then $Y_0 = 0$, Y_t is differentiable in t, $Y_t' > 0$ so once Y_t hits level 2 it will never hit level 2 again. Thus

$$\tau_2 := \inf\{t : Y_t = 2\}.$$

So we see that τ_2 is a stopping time.

(c) (5 points) Let τ_4 be a stopping time with respect to some filtration $\mathcal{F}(t)$ and $\tilde{\tau}_4$ be a random time such that $\tilde{\tau}_4 > \tau_4$. Can $\tilde{\tau}_4$ be a stopping time with respect to $\mathcal{F}(t)$? If yes provide an example, if no give an explanation.
Ans: It is possible that $\tilde{\tau}_4$ is a stopping time with respect to $\mathcal{F}(t)$. For example, let

$$\tau_4 := \inf\{t \geq 0 : W_t = 1\}$$
$$\tilde{\tau}_4 := \inf\{t \geq 0 : W_t = 2\},$$

where $W_0 = 0$, W_t is a Brownian motion. Then it is clear that both τ_4 and $\tilde{\tau}_4$ are stopping times, $\tilde{\tau}_4 > \tau_4$.

(d) (5 points) Let τ_5 be a stopping time with respect to some filtration $\mathcal{F}(t)$ and $\tilde{\tau}_5$ be a random time such that $\tilde{\tau}_5 < \tau_5$. Can $\tilde{\tau}_5$ be a stopping time with respect to $\mathcal{F}(t)$? If yes provide an example, if no give an explanation.

Ans: It is also possible that $\tilde{\tau}_5$ is a stopping time. We can use exactly the same example as above, except now we define

$$\tilde{\tau}_5 := \inf\{t \geq 0 : W_t = 1\}$$
$$\tau_5 := \inf\{t \geq 0 : W_t = 2\}.$$

2. (20 points) Let S_t follow the Black-Scholes dynamics under a risk neutral measure $\tilde{P}$:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

Let L_1, L_2 be two positive constants such that $L_1 < S_0 < L_2$. Consider an option on S_t that pays at time T

$$V_T = S_T \mathbf{1}_{\{\max_{0\leq u\leq T} S_u \leq L_2\}} \mathbf{1}_{\{\min_{0\leq u\leq T} S_u \geq L_1\}}.$$

In other words, this is a two barrier strike out option on S with barriers L_1 and L_2. If S_t ever gets below L_1 or above L_2 on the time interval $[0, T]$ then the option becomes worthless. Otherwise the option holdes gets paid S_T at time T. Write down the PDE for V_t. You do not have to show how you derive the PDE.

Ans:

The PDE is:

$$\begin{aligned}
-ru + u_t + rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx} &= 0, \text{ for } 0 \leq t \leq T, L_1 < x < L_2 \\
u(t, L_1) = u(t, L_2) &= 0 \\
u(T, x) &= x, L_1 < x < L_2.
\end{aligned}$$

To see why this is the case, let $\tau := \inf\{t \geq 0, S_t \leq L_1 \text{ or } S_t \geq L_2\}$ and recognize that τ is a stopping time and repeat exactly the same argument as the case with one barrier up and out option.

3. Let $\widetilde{P}$ be a risk neutral measure. Let S_t have the dynamics of geometric Brownian motion under $\widetilde{P}$:

$$dS_t = (0.5)S_t dt + \sqrt{2} S_t d\widetilde{W}_t.$$

Consider a perpetual American option on S_t with payoff function $g(S_t)$, where $g(x) = \big(K - \sqrt{x}\big)^+$ and K is a positive constant. Let V_t be the value of this American option at time t, assuming it has not been exercised.

(a) (10 points) By the Markov property of S_t, there is a bounded function $v(x)$ such that $v(S_t) = V_t$. State the linear complimentarity equations for $v(x)$. (You only need to state what they are; and do not have to justify).

The linear complimentarity equations are:

$$\begin{aligned} v(x) &\geq (K-\sqrt{x})^+ \\ -(0.5)v + v_x(0.5)x + v_{xx}x^2 &\leq 0 \\ -(0.5)v + v_x(0.5)x + v_{xx}x^2 &= 0 \text{ on } v(x) > (K-\sqrt{x})^+. \end{aligned}$$

(b) (10 points) Assuming that the continuation region has the form $C = \left\{x > L^*\right\}$ and exercise region has the form $E = \left\{0 \leq x \leq L^*\right\}$ where $L^* \leq K^2$. By using the smooth pasting principle, solve for L^* and the solution $v(x)$ of the above linear complimentarity equations. (You don't have to verify that $v(x)$ satisfy the linear complimentarity equations).

Ans:

The general solution to

$$-(0.5)v + v_x(0.5)x + v_{xx}x^2 = 0$$

is

$$v(x) = Ax^{-0.5} + Bx.$$

Since $v(x)$ is bounded for $x > 0$, $v(x) = Ax^{-0.5}$. So the form of the solution is

$$\begin{aligned} v(x) &= K - \sqrt{x}, 0 < x \leq L^* \\ &= Ax^{-0.5}, L^* < x. \end{aligned}$$

The smooth pasting conditions give

$$\begin{aligned} v(L^*) &= A(L^*)^{-0.5} = K - \sqrt{L^*} \\ v_x(L^*) &= -0.5A(L^*)^{-1.5} = -\frac{1}{2\sqrt{L^*}}. \end{aligned}$$

Solving for the above system gives

$$\begin{aligned} A &= L^* \\ L^* &= \left(\frac{K}{2}\right)^2. \end{aligned}$$

Thus

$$\begin{aligned} v(x) &= K - \sqrt{x}, 0 < x \leq \left(\frac{K}{2}\right)^2 \\ &= \frac{K^2}{4}x^{-.5}, \left(\frac{K}{2}\right)^2 < x. \end{aligned}$$

4. (20 points) Consider a model for the domestic market with a risky asset S_t and a foreign exchange rate Q_t. We assume the domestic market is the US market and the foreign market is the Euro market. They have the following dynamics under the domestic risk neutral measure $\widetilde{P}$:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_1 S_t d\widetilde{W}_t \\ dQ_t &= (r - r^f)Q_t dt + \sigma_2 Q_t d\widetilde{W}_t, \end{aligned}$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{P}$. All parameters are constants. To emphasize, the units of S_t, Q_t are in dollars.

Consider a call option on S_t with strike K where K is denominated in Euro:

$$V_T = (S_T - KQ_T)^+.$$

Note that V_T is denominated in dollars. Find an explicit formula for V_0, utilizing Black-Scholes formula (this is not a straightforward application of Black-Scholes formula. You will need to think about how to convert this problem into a framework that is appropriate for Black-Scholes). Your answer will involve $r, r^f, \sigma_1, \sigma_2, S_0, Q_0, K, T$.

Ans: Let $V_t^f = \frac{V_t}{Q_t}$ be the value of this option in Euro. Also denoting $S_t^f = \frac{S_t}{Q_t}$ be the value of the underlying asset in Euro then

$$V_T^f = (S_T^f - K)^+.$$

From the lecture we have under the foreign risk neutral measure $\widetilde{P}^f$

$$dS_t^f = r^f S_t^f dt + (\sigma_1 - \sigma_2) S_t^f d\widetilde{W}_t^f.$$

Thus V_0^f can be evaluated using the Black-Scholes formula:

$$V_0^f = S_0^f N(d+) - Ke^{-r^f T} N(d-),$$

where

$$d\pm = \frac{[r^f \pm \frac{1}{2}(\sigma_1 - \sigma_2)^2]T - \log(K/S_0^f)}{(\sigma_1 - \sigma_2)\sqrt{T}}$$

and $S_0^f = \frac{S_0}{Q_0}$. Finally $V_0 = V_0^f Q_0$.

5. Consider a model for the domestic market with a risky asset S_t and a foreign exchange rate Q_t. They have the following dynamics under the physical measure P (Note: not under a risk neutral measure) :

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dQ_t &= \alpha Q_t dt, \end{aligned}$$

where W_t is a Brownian motion. μ, α, σ are non-zero constants. Note that this implies the foreign exchange rate is deterministic. We also assume the domestic risk free rate r and the foreign risk free rate r^f are constants. In this problem, you can feel free to make any assumption about the relations between α, r, r^f that you like (for example $r < \alpha < r^f$ or $r + a < r^f$ etc.)

(a) (5 points) Does a domestic risk neutral measure exist? If yes explain how we can find it. If no explain why not.

Ans: Note that $N_t^f = Q_t e^{r^f t} = Q_0 e^{(\alpha + r^f)t}$. So $e^{-rt} N_t^f = Q_0 e^{(\alpha + r^f - r)t}$ is a strictly increasing or decreasing process (corresponding to $\alpha + r^f - r > 0$ or $\alpha + r^f - r < 0$). So a domestic risk neutral measure does NOT exist unless $\alpha + r^f - r = 0$. In that case, $e^{-rt} N_t^f = Q_0$ for all t so it is automatically a martingale. We define the risk neutral measure by using Girsanov theorem on the process S_t as usual.

(b) (5 points) Does a foreign risk neutral measure exist? If yes explain how we can find it. If no explain why not.

If the foreign risk neutral measure exists, then the domestic money market, denominated in foreign currency, discounted using the foreign risk free rate:

$$e^{-r^f t} \frac{1}{Q_t} e^{rt} = \frac{1}{Q_0} e^{(r - r^f - \alpha)}$$

must be a martingale under it. But this only happens if $r = r^f + \alpha$ as in part a. If so then we can define the foreign risk neutral measure as we did in class (actually $\tilde{P}^f = \tilde{P}$ in this case). If $r \neq r^f + \alpha$ then a foreign risk neutral measure does not exist.

(c) (10 points) Does an arbitrage opportunity exist in this market? If yes construct it. If no explain why not.

If $r = r^f + \alpha$ then an arbitrage opportunity does not exist. Otherwise there is an arbitrage opportunity. For example, suppose $r > r^f + \alpha$. We sell 1 Euro zero coupon bond, with maturity T in the Euro market for the price $e^{-r^f T}$ at time 0. We use this to invest in the US money market. At time T we receive $Q_0 e^{(-r^f + r)T}$. Note that this is worth more than 1 Euro at time T since $r - r^f > \alpha$ so $Q_0 e^{(-r^f + r)T} > Q_0 e^{\alpha T} = Q_T$. We only have to pay 1 Euro to the bond buyer so we have made a riskless profit.