Math 622
Spring 2016
Midterm exam 1
3/2/16

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- You can use the following moment generating functions in the exam:

$$\begin{aligned} E(e^{tN(\mu,\sigma^2)}) &= e^{t\mu+\frac{1}{2}t^2\sigma^2} \\ E(e^{t \text{ Poisson } (\lambda)}) &= e^{\lambda(e^t-1)}. \end{aligned}$$

Problem	Points	Score
1	20	
2	20	
3	10	
4	20	
5	20	
6	10	
Total:	100	

1. Let $Y_i, i = 1, 2, \cdots$ be independent, identically distributed $N(0, \sigma^2)$. Let N_t be a Poisson process with rate λ. Also suppose that $Y_i, i = 1, 2, \cdots$ are independent of N_t.

   (a) (10 points) Compute $E(e^{uY_{N_t}})$ where u is an arbitrary real number.

   Ans: Conditioning on N_t it is clear that

   $$E(e^{uY_{N_t}}|N_t = k) = E(e^{uY_k}|N_t = k) = E(e^{uY_k}) = e^{\frac{1}{2}u^2\sigma^2},$$

   where the second equality follows from the independence between Y_k and N_t. Thus $E(e^{uY_{N_t}}) = e^{\frac{1}{2}u^2\sigma^2}$ by the fact that $E(E(X|Y)) = E(X)$.

   (b) (10 points) Is Y_{N_t} is independent of N_t? If no explain. If yes prove your answer by establishing that $E(e^{u_1 Y_{N_t} + u_2 N_t}) = E(e^{u_1 Y_{N_t}})E(e^{u_2 N_t})$.

   Ans: They are independent. Arguing similarly to the above we have

   $$E(e^{u_1 Y_{N_t} + u_2 N_t}|N_t = k) = e^{u_2 k} E(e^{u_1 Y_k}|N_t = k) = e^{\frac{1}{2}u_1^2\sigma^2} e^{u_2 k},$$

   by the independence of Y_k and N_t. Thus generally

   $$E(e^{u_1 Y_{N_t} + u_2 N_t}|N_t) = e^{\frac{1}{2}u_1^2\sigma^2} e^{u_2 N_t}.$$

   Together with part a, it follows easily that $E(e^{u_1 Y_{N_t} + u_2 N_t}) = E(e^{u_1 Y_{N_t}})E(e^{u_2 N_t})$.

2. (20 points) Recall the Black-Scholes formula:

$$E\Big[e^{-rt}(xe^{(r-1/2\sigma^2 t)+\sigma W_t} - K)^+\Big] = xN(d+) - Ke^{-rt}N(d-),$$

where

$$d\pm = \frac{(r \pm 1/2\sigma^2)t - \log(K/x)}{\sigma\sqrt{t}}.$$

Now let N_t be a Poisson process with rate λ and W_t be a Brownian motion defined on the same probability space. Compute

$$E\Big[(N_t e^{N_t W_t} - K)^+\Big|N_t\Big].$$

Answer: N_t, W_t are independent. Now define

$$f(x) = E\Big[(xe^{xW_t} - K)^+\Big].$$

This is in the context of Black-Scholes formula with $\sigma = x$ and we require $r - \frac{1}{2}\sigma^2 = 0$ or $r = \frac{1}{2}x^2$. With these choices of r, σ we have

$$f(x) = e^{rt}E\Big[e^{-rt}(xe^{\sigma W_t} - K)^+\Big].$$

Thus applying B-S formula we have

$$f(x) = e^{rt}\Big(xN(d+) - Ke^{-rt}N(d-)\Big),$$

where

$$d\pm \quad = \quad \frac{(r \pm \frac{1}{2}\sigma^2)t - \log(K/x)}{\sigma\sqrt{t}}.$$

Now by the independence lemma,

$$E\Big[(N_t e^{N_t W_t} - K)^+\Big|N_t\Big] = f(N_t).$$

3. (10 points) Let N^1, N^2 be two independent Poisson processes with rate λ_1, λ_2 respectively. Is the process

$$X_t := \int_0^t N_u^2 d(N_u^1 - \lambda_1 u)$$

a martingale? (Note: it is N_u^2 in the integrand, not N_{u-}^2). Justify your answer.

Ans: X_t is a martingale since actually

$$X_t := \int_0^t N_{u-}^2 d(N_u^1 - \lambda_1 u).$$

This follows from the fact that

$$\int_0^t N_u^2 d(N_u^1 - \lambda_1 u) = \int_0^t N_{u-}^2 d(N_u^1 - \lambda_1 u) + \int_0^t \Delta N_u^2 dN_u^1 - \lambda_1 \int_0^t \Delta N_u^2 du$$

and $\int_0^t \Delta N_u^2 dN_u^1 = 0$ since N^1, N^2 do not jump at the same time; $\int_0^t \Delta N_u^2 du = 0$ since N^2 only jumps finitely many times on the interval $[0, t]$.

4. Let S_t have the following dynamics

$$dS_t = rS_t dt + \sigma S_t d(N_t - \lambda t),$$

where N_t is a Poisson λ process. Again, note that it is $\sigma S_t d(N_t - \lambda t)$, not $\sigma S_{t-} d(N_t - \lambda t)$ in the model.

(a) (10 points) Find an explicit formula for S_t, given S_0.

Ans: Note that at the jump point of N_t:

$$S_t - S_{t-} = \sigma S_t.$$

That is $S_t = \frac{1}{1-\sigma} S_{t-}$. Thus

$$S_t = S_0 e^{(r-\lambda\sigma)t} \left(\frac{1}{1-\sigma}\right)^{N_t}.$$

(b) (10 points) Prove that $e^{-rt}S_t$ is not a martingale.

$$e^{-rt}S_t = S_0 e^{-\lambda\sigma t} \left(\frac{1}{1-\sigma}\right)^{N_t} = S_0 e^{-\lambda\sigma t - N_t \log(1-\sigma)}.$$

It is enough to show that $E(e^{-rt}S_t) \neq S_0$. Applying the moment generating function of a Poisson

$$E(e^{-N_t \log(1-\sigma)}) = e^{\lambda t(\frac{1}{1-\sigma} - 1)} \neq e^{\lambda\sigma t}.$$

5. Let S_t have the following dynamics

$$dS_t = rS_t dt + \sigma S_t d(N_t - \lambda t),$$

where N_t is a Poisson λ process. Again, note that it is $\sigma S_t d(N_t - \lambda t)$, not $\sigma S_{t-} d(N_t - \lambda t)$ in the model.

(a) (10 points) Let $V_t = E(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t)$. Prove that $e^{-rt}V_t$ is a martingale.
Ans: Since $e^{-rt}V_t = E(e^{-rT}V_T|\mathcal{F}_t)$ it is clear that $e^{-rt}V_t$ is a martingale.

(b) (10 points) It can be showed that we can find a function $u(t,x)$ such that $u(t,S_t) = V_t$. Find the partial differential difference equation that $u(t,x)$ satisfies.
Ans:

$$d(e^{-rt}u(t,S_t)) = e^{-rt}\Big[(-ru + u_t + u_x(r - \sigma\lambda)S_t dt + u(t,S_t) - u(t,S_{t-})\Big].$$

Now

$$u(t,S_t) - u(t,S_{t-}) = [u(t, S_{t-}\frac{1}{1-\sigma}) - u(t,S_{t-})]\Delta N_t,$$

it follows that the PDDE is

$$\begin{aligned} -ru + u_t + u_x(r - \sigma\lambda)x + \lambda(u(t, x\frac{1}{1-\sigma}) - u(t,x)) &= 0 \\ u(T,x) &= (x-K)^+. \end{aligned}$$

6. (10 points) Let M_t be a martingale with jumps and X_t a *right continuous* jump process with left limits. Mr. Pham says that $\int_0^t X_u dM_u$ is *not* a martingale because X_t is not left continuous. Is Mr. Pham right or wrong? Explain your answer.

   Ans: Mr. Pham is wrong. Problem 3 provides a counter example.

   Alternative (wrong) answer: Mr. Pham is always right !