Math 622
Spring 2014
Final Exam
05/09/2016

Name (Print): __________________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Black-Scholes formula:

$$E\Big[e^{-rt}(xe^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} - K)^+\Big] = xN(d+) - Ke^{-rt}N(d-),$$

where

$$d\pm = \frac{(r \pm \frac{1}{2}\sigma^2)t - \log(K/x)}{\sigma\sqrt{t}}.$$

Problem	Points	Score
1	40	
2	40	
3	40	
4	40	
5	20	
6	20	
Total:	200	

1. Let the forward rate $f(t,T)$ have the following dynamics under the physical measure P:

$$df(t,T) = \alpha(t,T)dt + \sigma W(t),$$

where σ is a *deterministic constant*.

(a) (10 points) What is the dynamics of $f(t,T)$ under the risk neutral measure $\tilde{P}$? (You only need to state, and don't have to derive the result.)
Ans: Let $\sigma^*(t,T) = \int_t^T \sigma du = \sigma(T-t)$, we have

$$\begin{aligned} df(t,T) &= \sigma\sigma^*(t,T)dt + \sigma d\tilde{W}(t) \\ &= \sigma^2(T-t)dt + +\sigma d\tilde{W}(t). \end{aligned}$$

(b) (10 points) Suppose $f(0,T) = \frac{1}{T+1}, T \geq 0$. Find an explicit formula for $f(t,T)$ in terms of $\sigma, \tilde{W}, t, T$.
Ans:

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t df(u,T) \\ &= \frac{1}{T+1} - \frac{\sigma^2}{2}\big((T-t)^2 - T^2\big) + \sigma\tilde{W}(t). \end{aligned}$$

(c) (10 points) Find the dynamics of $R(t)$, the short rate, under $\tilde{P}$.
Ans:

$$R(t) = f(t,t) = \frac{1}{t+1} + \frac{\sigma^2 t^2}{2} + \sigma\tilde{W}(t).$$

Therefore

$$dR(t) = \Big(-\frac{1}{(t+1)^2} + \sigma^2 t\Big)dt + \sigma d\tilde{W}(t).$$

(d) (10 points) Find $\tilde{E}\big(R(T)\big)$, $\tilde{\text{Var}}\ \big(R(T)\big)$ where $\tilde{\text{Var}}$ means we compute the variance under the risk neutral measure $\tilde{P}$.
Ans: From the above, we have

$$\begin{aligned} \tilde{E}\big(R(T)\big) &= \frac{1}{T+1} + \frac{\sigma^2 T^2}{2} \\ \tilde{\text{Var}}\ \big(R(T)\big) &= \sigma^2 T. \end{aligned}$$

2. (40 points) Consider a market with two assets S^1, S^2 that have the following dynamics under a risk neutral measure $\widetilde{P}$

$$
\begin{aligned}
dS_t^1 &= rS_t^1 dt + \sigma^1 S_t^1 d\widetilde{W}_t \\
dS_t^2 &= rS_t^2 dt + \sigma^2 S_t^2 d\widetilde{W}_t,
\end{aligned}
$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{P}$ and σ^1, σ^2, r are constants. Consider an option V that pays

$$V_T = (S_T^1 - S_T^2)^+$$

at time T. Find an explicit formula for V_0 utilizing Black-Scholes formula. (This is again not a straightforward application of Black-Scholes . You will need to think about how to transform this problem into one that is appropriate for Black-Scholes framework).

Ans: This is exactly the same as problem 4 in Midterm 2 with S^1 being S and S^2 being N^f, the foreign money market and $K = 1$. More specifically, using S^2 as numéraire, the pricing equation becomes

$$V_0^{(S^2)} = \widetilde{E}^{(S^2)}({S^1}_T^{(S^2)} - 1)^+,$$

where $V_0^{(S^2)} = \frac{V_0}{S_0^2}$ and ${S^1}_T^{(S^2)} = \frac{S_T^1}{S_T^2}$. The dynamics of ${S^1}_t^{(S^2)}$ under $\widetilde{P^{(S^2)}}$ is

$$d{S^1}_t^{(S^2)} = {S^1}_t^{(S^2)}(\sigma^1 - \sigma^2) d\widetilde{W}_t^{(S^2)}.$$

Thus Black-Scholes formula gives

$$V_0^{(S^2)} = {S^1}_0^{(S^2)} N(d+) - N(d-),$$

where

$$d\pm = \frac{\pm\frac{1}{2}(\sigma^1 - \sigma^2)^2 T - \log(1/{S^1}_0^{(S^2)})}{(\sigma^1 - \sigma^2)\sqrt{T}}.$$

Or

$$V_0 = S_0^1 N(d+) - S_0^2 N(d-),$$

where

$$d\pm = \frac{\pm\frac{1}{2}(\sigma^1 - \sigma^2)^2 T - \log(\frac{S_0^2}{S_0^1})}{(\sigma^1 - \sigma^2)\sqrt{T}}.$$

3. Consider the following dynamics for the forward LIBOR rate $L_\delta(t,T)$ under the forward measure $\widetilde{P}^{T+\delta}$:

$$dL_\delta(t,T) = \sqrt{2t} L_\delta(t,T) d\widetilde{W}_t^{T+\delta}, 0 \le t \le T.$$

In this problem, for convenience of computation we suppose $T = \delta = 1$.

(a) (20 points) Find an explicit formula for $L_1(1,1)$, assuming $L_1(0,1)$ is given.

$$L_1(1,1) = L_1(0,1) \exp\Big(-\frac{1}{2} \int_0^1 2u du + \int_0^1 \sqrt{2u} d\widetilde{W}_u^2 \Big).$$

(b) (20 points) Consider the caplet that pays $V_2 = (L_1(1,1) - K)^+$ at time 2 with strike K. Find an explicit formula for V_0 using Black-Scholes formula.
From part a), $L_1(1,1)$ has log-normal distribution. That is $L_1(1,1) = L_1(0,1)e^X$ where X is a Normal(-1/2, 1). The pricing formula for the caplet is

$$V_0 \quad = \quad B(0,2)\widetilde{E}^2((L_1(1,1) - K)^+).$$

Thus applying the Black-Scholes formula we have

$$V_0 \quad = \quad B(0,2)\Big[e^{rT} x N(d+) - K N(d-)\Big],$$

where

$$d\pm = \frac{(r \pm \frac{1}{2}\sigma^2)T - \log(K/x)}{\sigma\sqrt{T}}$$

and (matching coefficients) $x = L_1(0,1), T = 2$. σ needs to satisfy $\sigma^2 T = 1$ thus $\sigma = 1/\sqrt{2}$. r needs to satisfy

$$(r - \frac{1}{2}\sigma^2)T = -1/2.$$

That is $r = 0$.

4. (40 points) Consider the following model for the forward rate under a risk neutral measure $\widetilde{P}$:

$$f(t,T) = TN(t),$$

where $N(t)$ is a Poisson($\widetilde{\lambda}$) process under $\widetilde{P}$.

Recall that under the risk neutral measure $\widetilde{P}$, $e^{-\int_0^t R_u du} B(t,T)$ is a martingale for any $B(t,T)$, where R_t is the risk free short rate. Also recall that the following relationship holds:

$$B(t,T) = e^{-\int_t^T f(t,u)du}.$$

Find the relationship between $R_t, f(t,T)$ and $\widetilde{\lambda}$ in this model.

We have

$$B(t,T) = e^{-N(t)\frac{T^2-t^2}{2}}.$$

At the jump point of $N(t)$:

$$B(t,T) - B(t-,T) = e^{-N(t)\frac{T^2-t^2}{2}} - e^{-N(t-)\frac{T^2-t^2}{2}} = e^{-N(t-)\frac{T^2-t^2}{2}}(e^{-\frac{T^2-t^2}{2}} - 1) = B(t-,T)(e^{-\frac{T^2-t^2}{2}} - 1).$$

Thus

$$\begin{aligned} dB(t,T) &= B(t,T)tN(t)dt + B(t-,T)(e^{-\frac{T^2-t^2}{2}} - 1)dN(t) \\ &= [(e^{-\frac{T^2-t^2}{2}} - 1)\widetilde{\lambda} + tN(t)]B(t,T)dt + B(t-,T)(e^{-\frac{T^2-t^2}{2}} - 1)d(N(t) - \widetilde{\lambda}t). \end{aligned}$$

The drift term in $B(t,T)$ under the risk neutral measure must be $R(t)$. Thus we have

$$(e^{-\frac{T^2-t^2}{2}} - 1)\widetilde{\lambda} + tN(t) = R(t)$$

or

$$(e^{-\frac{T^2-t^2}{2}} - 1)\widetilde{\lambda} + t\frac{f(t,T)}{T} = R(t).$$

5. Let V_t^{EC} denote the value of a European call option on an underlying asset S_t with expiration T and strike K. Let V_t^{EP} denote the value of the corresponding European put option on the same asset with same strike and expiration. Suppose also the risk free interest rate is a constant $r > 0$.

   (a) (10 points) Express $V_t^{EC} - (S_t - K)$ in terms of V_t^{EP}, K and r.
   By the Put-Call parity : $V_t^{EC} - V_t^{EP} = S_t - e^{-r(T-t)}K$. Thus

   $$V_t^{EC} - (S_t - K) = V_t^{EP} + (K - e^{-r(T-t)}K).$$

   (b) (10 points) Use your answer in part (a) to conclude that it is never optimal to exercise an American call option before T. Hence the value of American call option and European call option are equal.

   The RHS of the above equation is always positive for $t < T$. Thus the exercise value of the American call option, $S_t - K$ (when $S_t \geq K$ of course) is always strictly less than V_t^{EC}, which is less than or equal to the value of the American call option. Thus it is not optimal to exercise for $t < T$.

6. Recall the well-known Black-Scholes formula: if the risk neutral dynamics of the asset S_t is

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

then $V_0 = S_0 N(d+) - Ke^{-rT} N(d-)$ where

$$d\pm = \frac{(r \pm \frac{1}{2}\sigma^2)t - \log(K/x)}{\sigma\sqrt{t}}.$$

(a) (10 points) $N(d-)$ can be interpreted as $\widetilde{P}(E)$ where $\widetilde{P}$ is the risk neutral measure and E is a random event. What is this event E?

By risk neutral pricing

$$\begin{aligned} V_0 &= \widetilde{E}(e^{-rT}(S_T - K)^+) \\ &= \widetilde{E}(e^{-rT}(S_T - K)\mathbf{1}_{S_T \geq K}) \\ &= \widetilde{E}(e^{-rT} S_T \mathbf{1}_{S_T \geq K}) - Ke^{-rT}\widetilde{P}(S_T \geq K). \end{aligned}$$

Thus E is the event $S_T \geq K$.

(b) (10 points) Similarly $N(d+)$ can be interpreted as $\widetilde{P}^{(N)}(E)$ where $\widetilde{P}^{(N)}$ is the measure associated with a certain numéraire N and E is the same event as in part (a). What is this numéraire N?

$$\widetilde{E}(e^{-rT} S_T \mathbf{1}_{S_T \geq K}) = S_0 \widetilde{E}(\frac{e^{-rT} S_T}{S_0} \mathbf{1}_{S_T \geq K}) = S_0 \widetilde{P}^{(S)}(S_T \geq K).$$

Thus the numéraire is S itself !