Homework 9 (Due 4/20/2016)

Math 622

April 7, 2016

1. Suppose the risk-neutral model for a risky asset $S(t)$, $t \le 2$, and a zero-coupon bond $B(t,2)$ maturing at $T = 2$ is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + [5-t]S(t)\,d\widetilde{W}(t), \quad t \le 2; & (1)\\ dB(t,2) &= R(t)B(t,2)\,dt + (2-t)B(t,2)\,d\widetilde{W}(t), \quad t \le 2. & (2) \end{aligned}$$

Let $S^{(2)}(t) = \dfrac{S(t)}{B(t,2)}$ be the price of the risky asset denominated in the zero-coupon bond price. This is the $\{T = 2\}$-forward price of $S(t)$, which is denoted by $\mathrm{For_S(t,2)}$ in Shreve, but the notation $S^{(2)}$ is simpler to write!

Let $\widetilde{\mathbf{P}}^{(2)}$ denote the risk-neutral measure for the numéraire $B(t,2)$.

The first part of this problem will give you practice in understanding and applying Theorems 9.2.1 and 9.2.2 of Shreve. The second part is based on section 9.4.3; see the class notes, *Notes to Lecture 10*, the last section, for a more straightforward derivation of Theorem 9.4.2, which is relevant to part b)

(a) Show how to define $\widetilde{W}^{(2)}$ so that it is a Brownian motion under $\widetilde{\mathbf{P}}^{(2)}$. Write down a stochastic differential equation for $S^{(2)}$ under the measure $\widetilde{\mathbf{P}}^{(2)}$ using $\widetilde{W}^{(2)}$.

(b) Find an explicit formula for $V(0) = \tilde{E}\big[D(2)(S(2)-K)^+ | S(0) = s_0\big]$, in terms of s_0 and $B(0,2)$.

2. In section 9.4.3, Shreve derives a formula for the price of a call option when the risk-free rate is random by assuming that the T-forward price of the underlying has constant volatility σ. This model is written as equation (9.4.8).

Assume that under the original risk neutral model, the zero-coupon bond price satisfies equation (9.4.4). Derive the stochastic differential equation for $S(t)$ valid under the original risk-neutral measure implied by the model (9.4.8), and determine how the volatility of $S(t)$ is related to the volatility of the zero-coupon bond price. (You might ponder whether this is a realistic scenario or not!)

(This is another exercise on Theorem 9.2.2 —you have to use it in reverse.)

3. Shreve, Exercise 9.3. (Yet another exercise using Theorems 9.2.1 and 9.2.2.)

4. Shreve, Exercise 9.5.