Homework 10

Math 622

April 21, 2016

1. Suppose the risk-neutral model for a risky asset $S(t)$, $t \le 2$, and a zero-coupon bond $B(t,2)$ maturing at $T=2$ is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + [5-t]S(t)\,d\widetilde{W}(t), \quad t \le 2; \qquad (1) \\ dB(t,2) &= R(t)B(t,2)\,dt + (2-t)B(t,2)\,d\widetilde{W}(t), \quad t \le 2. \qquad (2) \end{aligned}$$

Let $S^{(2)}(t) = \dfrac{S(t)}{B(t,2)}$ be the price of the risky asset denominated in the zero-coupon bond price. This is the $\{T=2\}$-forward price of $S(t)$, which is denoted by $\mathrm{For_S(t,2)}$ in Shreve, but the notation $S^{(2)}$ is simpler to write!

Let $\widetilde{\mathbf{P}}^{(2)}$ denote the risk-neutral measure for the numéraire $B(t,2)$.

(a) Show how to define $\widetilde{W}^{(2)}$ so that it is a Brownian motion under $\widetilde{\mathbf{P}}^{(2)}$. Write down a stochastic differential equation for $S^{(2)}$ under the measure $\widetilde{\mathbf{P}}^{(2)}$ using $\widetilde{W}^{(2)}$.

Ans: The change of measure to the $\widetilde{\mathbf{P}}^{(2)}$ measure is defined by

$$\frac{d\widetilde{\mathbf{P}}^{(2)}}{d\widetilde{\mathbf{P}}} = \frac{D(2)B(2,2)}{B(0,2)} = e^{\int_0^2 (2-s)\,d\widetilde{W}(s) - (1/2)\int_0^2 (2-s)^2\,ds}.$$

Therefore, by Girsanov's theorem,

$$\widetilde{W}^{(2)}(t) = \widetilde{W}(t) - \int_0^t (2-s)\,ds$$

is a martingale under $\widetilde{\mathbf{P}}^{(2)}$.

By Theorem 9.2,

$$dS^{(2)}(t) = S^{(2)}(t)[(5-t)-(2-t)]\,d\widetilde{W}^{(2)}(t) = 3S^{(2)}(t)\,d\widetilde{W}^{(2)}(t).$$

(b) Find an explicit formula for $V(0) = \tilde{E}\Big[D(2)(S(2)-K)^+ | S(0)=s_0\Big]$, in terms of s_0 and $B(0,2)$.

Ans:

$$\frac{V(0)}{B(0,2)} = V^{(2)}(0) = \tilde{E}^{(2)}\Big[\frac{(S(2)-K)^+}{B(2,2)}\Big|S(0)=s_0\Big] = \tilde{E}^{(2)}\Big[(S^{(2)}(2)-K)^+\Big|S^{(2)}(0)=s_0/B(0,2)\Big],$$

because $B(2,2) = 1$ and hence $S(2) = S^{(2)}(2)$. But from part (a), $S^{(2)}$ follows a standard Black-Scholes model with $r = 0$ and $\sigma = 3$. Thus by using the Black-Scholes formula,

$$V(0) = B(0,2)\left[\frac{s_0}{B(0,2)}N(d_+(s_0/B(0,2),K)) - KN(d_-(s_0/B(0,2),K))\right],$$

where

$$d_\pm(x,K) = \frac{1}{3\sqrt{2}}\left[\ln\frac{x}{K} \pm 9\right].$$

2. In section 9.4.3, Shreve derives a formula for the price of a call option when the risk-free rate is random by assuming that the T-forward price of the underlying has constant volatility σ. This model is written as equation (9.4.8).

Assume that under the original risk neutral model, the zero-coupon bond price satisfies equation (9.4.4). Derive the stochastic differential equation for $S(t)$ valid under the original risk-neutral measure implied by the model (9.4.8), and determine how the volatility of $S(t)$ is related to the volatility of the zero-coupon bond price. (You might ponder whether this is a realistic scenario or not!)

(This is another exercise on Theorem 9.2.2 —you have to use it in reverse.)

Ans: Suppose $dS(t) = R(t)S(t)\,dt + \overline{\sigma}(t)S(t)\,d\widetilde{W}(t)$ under the original risk-neutral measure. According to Theorem 9.2, if the zero-coupon bond price satisfies (9.4.4), namely $d[D(t)B(t,T)] = -D(t)B(t,T)\sigma^*(t,T)\,d\widetilde{W}(t)$, then $-\sigma^*(t,T)$ is playing the role of ν in Theorem 9.2 and

$$d\text{For}_S(t,T) = \text{For}_S(t,T)[\overline{\sigma}(t) + \sigma^*]\,d\widetilde{W}^T(t).$$

Here $\widetilde{W}^T$ is the Brownian motion under the T-forward measure as define in section 9.4; $\widetilde{W}^T(t) = \widetilde{W}(t) + \int_0^t \sigma^*(u,T)\,du$, for $0 \le t \le T$. But we are requiring that

$$d\text{For}_S(t,T) = \sigma_T \text{For}_S(t,T)\,d\widetilde{W}^T(t),$$

for some constant σ. (We have put a subscript T on σ because the model depends on T. It follows that, $\overline{\sigma(t)} = \sigma_T - \sigma^*(t,T)$, that is

$$dS(t) = R(t)S(t)\,dt + [\sigma_T - \sigma^*(t,T)]\,S(t)\,d\widetilde{W}(t).$$

This says that the volatility of the asset price is a constant translate of the volatility of the zero-coupon bond price. This seems very unrealistic. For example, if true, it should for different T, which would then imply that the volatilities of zero-coupon bonds of different maturities are all translates of one another, which also seems unlikely.

3. Shreve, Exercise 9.3. (Yet another exercise using Theorems 9.2.1 and 9.2.2.)

Ans: (i) By Itô's rule,

$$d\frac{1}{N(t)} = (\nu^2 - r)\frac{1}{N(t)}\,dt - \nu\frac{1}{N(t)}\,d\widetilde{W}_3(t).$$

Thus $dS(t)\,d[1/N(t)] = -\sigma\nu[S(t)/N(t)]\,d\widetilde{W}_1(t)\,d\widetilde{W}_3(t) = -\sigma\nu\rho S^{(N)}(t)\,dt$. Then, using Itô's rule,

$$\begin{aligned} dS^{(N)}(t) &= \frac{1}{N(t)}dS(t) + S(t)d\frac{1}{N(t)} + dS(t)d[1/N(t)] \\ &= (\nu^2 - \sigma\nu\rho)S^{(N)}\,dt + \sigma S^{(N)}(t)\,d\widetilde{W}_1(t) - \nu S^{(N)}(t)\,d\widetilde{W}_3(t). \end{aligned}$$

Define $\gamma = \sqrt{\sigma^2 - 2\rho\sigma\nu + \nu^2}$ and

$$\widetilde{W}_4(t) = \frac{1}{\gamma}\left[\sigma\,d\widetilde{W}_1(t) - \nu\,d\widetilde{W}_3(t)\right].$$

Then $(dW_4(t))^2 = \frac{1}{\gamma^2}\left[\sigma^2(dW_1(t))^2 - 2\sigma\nu\,d\widetilde{W}_1(t)d\widetilde{W}_3(t) + \nu^2(d\widetilde{W}_3(t))^2\right] = \frac{\sigma^2-2\sigma\nu\rho+\nu^2}{\gamma^2}\,dt = dt$. By Lévy's theorem $\widetilde{W}_4$ is a Brownian motion. Using this in the equation given above for $S^{(N)}(t)$ leads to

$$dS^{(N)}(t) = (\nu^2 - \sigma\nu\rho)S^{(N)}\,dt + \gamma S^{(N)(t)}\,d\widetilde{W}_4(t).$$

(ii) Define $\widetilde{W}_2 = \frac{1}{\sqrt{1-\rho^2}}\left[\widetilde{W}_3 - \rho\widetilde{W}_1(t)\right]$. Then one can verify easily that $(d\widetilde{W}_2(t))^2 = dt$, so that $\widetilde{W}_2(t)$ is a Brownian motion, and $d\widetilde{W}_1(t)d\widetilde{W}_2(t) = 0$, so that $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent. Then $d\widetilde{W}_3(t) = \rho\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\,d\widetilde{W}_2(t)$, and

$$dN(t) = rN(t)\,dt + \nu N(t)\,\left[\rho\,d\widetilde{W}_2(t) + \sqrt{1-\rho^2}\,d\widetilde{W}_2(t)\right] \tag{3}$$

Let $\mathbf{v} = (\nu\rho, \nu\sqrt{1-\rho^2})$ for all t, and let $\widetilde{W}(t) = (\widetilde{W}_1(t), \widetilde{W}_2(t))$. Then it follows from equation (3) that

$$d(D(t)N(t)) = D(t)N(t)\,\mathbf{v}\cdot d\widetilde{W}(t).$$

Similarly $d(D(t)S(t)) = D(t)S(t)(\sigma, 0) \cdot d\widetilde{W}(t)$. Thus, by equation (9.2.9) in Theorem 9.2.2,

$$\begin{aligned} dS^{(N)}(t) &= S^{(N)}(t)\left[(\sigma, 0) - (\nu\rho, \nu\sqrt{1-\rho^2})\right] \cdot d\widetilde{W}(t) \\ &= S^{(N)}(t)\left[(\sigma - \nu\rho)\, d\widetilde{W}_1(t) - \nu\sqrt{1-\rho^2}\, d\widetilde{W}_2(t)\right] \end{aligned}$$

Let $v_1 = \sigma - \nu\rho$ and $v_2 = -\nu\sqrt{1-\rho^2}$. Then

$$v_1^2 + v_2^2 = \sigma^2 - 2\nu\sigma\rho + \nu^2\rho^2 + \nu^2(1-\rho^2) = \sigma^2 - 2\rho\sigma\nu + \nu^2.$$

4. Shreve, Exercise 9.5.

Ans: (i) The price S satisfies $dS(t) = rS(t)\,dt + \sigma_1 S(t)\,d\widetilde{W}_1(t)$, and the solution to this is $S(t) = S(0)\exp\{\sigma_1\widetilde{W}_1(t) + (r - \frac{1}{2}\sigma_1^2)t\}$.

(ii) The equation for Q is $dQ(t) = (r - r^r)Q(t)\,dt + \sigma_2\left[\rho\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\, d\widetilde{W}_2\right]$. We know (see equations (11), (12), and (13) in Lecture Notes 7) that this solution to this is

$$Q(t) = Q(0)\exp\left\{\int_0^t \nu \cdot\, d\widetilde{W}(t) + \int_0^t (r - r^f - \frac{1}{2}\|\nu\|^2)\,du\right\},$$

where $\nu = (\sigma_2\rho, \sigma_2\sqrt{1-\rho^2})$. One easily computes that $\|\nu\|^2 = \sigma_2^2$. Because ν is constant,

$$Q(t) = Q(0)\exp\left\{\sigma_2\rho\widetilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}\widetilde{W}_2(t) + (r - r^f - \frac{1}{2}\sigma_2^2)t\right\}.$$

(iii) Define $\widetilde{W}_4(t) = \frac{\sigma_1 - \sigma_2\rho}{\sigma_4}\widetilde{W}_1(t) - \frac{\sigma_2\sqrt{1-\rho^2}}{\sigma_4}\widetilde{W}_2(t)$, where $\sigma_4^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$. It is easily verified that $[d\widetilde{W}_4(t)]^2 = dt$ and hence that $\widetilde{W}_4$ is a Brownian motion. Clearly,

$$\sigma_4\, d\widetilde{W}_4(t) = (\sigma_1 - \sigma_2\rho)\, d\widetilde{W}_1(t) - \sigma_2\sqrt{1-\rho^2}\, d\widetilde{W}_2(t).$$

From parts (i) and (ii),

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)}\exp\left\{(\sigma_1 - \sigma_2\rho)\, d\widetilde{W}_1(t) - \sigma_2\sqrt{1-\rho^2}\, d\widetilde{W}_2(t) + (r^f + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)t\right\}$$

Define a so that

$$r - a - \frac{1}{2}\sigma_4^2 = -r^f + \frac{1}{2}(\sigma_2^2 - \sigma_1^2).$$

A calculation shows that $a = r - r^f - \sigma_2^2 + \rho\sigma_1\sigma_2$. The purpose of this definition is to write

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \exp\{\sigma_4 \widetilde{W}_4(t) + (r - a - \frac{1}{2}\sigma_4^2)t\}.$$

This is the solution to the Black-Scholes price equation with initial price $S(0)/Q(0)$, driving Brownian motion $\widetilde{W}_4$, volatility σ_4, and interest rate $r - a$.

(iv) Let $C(T - t, x, K, r, \sigma)$ be the price of a call option with strike K, interest rate r, and volatility σ. The formula for C, with different notation, is given in (4.5.22). As just pointed out $S(t)/Q(t)$ has the form of a Black-Scholes price. The value of a quanto at strike K is

$$V(t) = e^{-r(T-t)} \tilde{E}\left[\left(\frac{S(T)}{Q(T)} - K\right)^+ \,\Big|\, \mathcal{F}(t)\right]$$

Adjust this to $V(t) = e^{-at} e^{-(r-a)t} \tilde{E}\left[\left(\frac{S(T)}{Q(T)} - K\right)^+ \,\Big|\, \mathcal{F}(t)\right]$. Aside from the factor e^{-at} this is the price of a call with risk-free rate $r - a$ and volatitility σ_4. Hence,

$$V(t) = e^{-at} C(T - t, \frac{S(t)}{Q(t)}, K, r - a, \sigma_4),$$

and substituting this into (4.5.22) gives the formula shown in the problem statement.