Homework 9

Math 622

April 7, 2016

1. A submartingale is defined on page 74, of the text. Jensen's inequality is stated on page 18 and the conditional Jensen inequality is stated on page 70.

a) Show: if $X(t)$, $t \geq 0$, is a martingale, and if f is a convex function such that $E[|f(X(t))|] < \infty$ for all t, then $f(X(t))$ is a submartingale.

Ans: Let $\{X(t);\, t \geq 0\}$ be a martingale, let f be a convex function and let $Y(t) = f(X(t))$. Let $0 \leq s < t$. Then, by Jensen's inequality for conditional expectation,

$$Y(s) = f(X(s)) = f\Big(E[X(t)\big|\mathcal{F}(s)]\Big) \leq E\big[f(X(t))\big|\mathcal{F}(s)\big] = E[Y(t)\big|\mathcal{F}(s)].$$

This shows that $\{Y(t);\, t \geq 0\}$ is a submartingale.

b) Let S denote an asset price process. If the risk-free interest rate in a risk-neutral model is $r = 0$, and h is a convex payoff function, show that $h(S(t))$ is a submartingale. Show that the value of an American and European option expiring at T are the same, if the payoff function is h.

Ans: If the risk-free rate is 0, then $\{S(t);\, t \geq 0\}$ is itself a submartingale, and the value of an option with payoff h is $V(t) = \tilde{E}[h(S(T)\big|\mathcal{F}(t)]$. If h is convex, it follows from a) that $h(S(t))$ is a martingale. Then,

$$h(S(t)) \leq \tilde{E}[h(S(T))\big|\mathcal{F}(t)].$$

This says the the value of exercising at the terminal time is always at least as great as the value of immediate exercise. Hence, even if one has the option to exercise at any time, one can always do just as well by exercising at T. Therefore the values of the American and European option with payoff h are the same.

2. Let r be the risk-free rate, and consider a risk-neutral model,

$$\begin{aligned} dS_1(t) &= rS_1(t)\,dt + \sigma_1 S_1(t)\,d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\,dt + \sigma_2 S_2(t)\Big[d\widetilde{W}_1(t) + 2\,d\widetilde{W}_2(t)\Big], \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions. Let $g(x_1, x_2)$ be a bounded payoff function. Consider the American option with payoff function g and expiration $T < \infty$. Define,

$$\begin{aligned} &v(t, x_1, x_2) \\ &\quad = \sup\Big\{\tilde{E}[e^{-r\tau}g(S_1(\tau), S_2(\tau))\Big|S_1(t){=}x_1, S_2(t){=}x_2\Big];\ \tau \text{ is a stopping time, } t \le \tau \le T.\Big\} \end{aligned}$$

Thus $v(t, S_1(t), S_2(t))$ is the price of the American option at time t, assuming it has not been exercised yet.

Find the linear complementarity equations for $v(t, x_1, x_2)$. (Note, the time parameter t is a factor, as in the American option with finite expiration.) Hint: first set up the martingale and supermartingale conditions for characterizing v, by generalizing Theorem 3 of the *Notes to Lecture 8*.

Ans: Let r be the risk-free rate, and consider a risk-neutral model,

$$\begin{aligned} dS_1(t) &= rS_1(t)\,dt + \sigma_1 S_1(t)\,d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\,dt + \sigma_2 S_2(t)\Big[d\widetilde{W}_1(t) + 2\,d\widetilde{W}_2(t)\Big], \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions. Let $g(x_1, x_2)$ be a bounded payoff function. Consider the American option with payoff function g and expiration $T < \infty$. Define,

$$\begin{aligned} &v(t, x_1, x_2) \\ &\quad = \sup\Big\{\tilde{E}[e^{-r\tau}g(S_1(\tau), S_2(\tau))\Big|S_1(t){=}x_1, S_2(t){=}x_2\Big];\ \tau \text{ is a stopping time, } t \le \tau \le T.\Big\} \end{aligned}$$

We shall derive the linear complimentarity conditions for v.

v must be a bounded function satisfying $v(t, x_1, x_2) \ge g(x_1, x_2)$ for all $x_1, x_2 \ge 0$, and $0 \le t \le T$. We want v also to satisfy the properties: $e^{-rt}v(t, S_1(t), S_2(t))$, is a supermartingale for $0 \le t \le T$; and $e^{-r(t\wedge\tau^*)}v(t \wedge \tau*, S_1(t \wedge \tau^*), S_2(t \wedge \tau^*))$, is a martingale for $0 \le t \le T$, where $\tau^* = \min\{t; v(t, S_1(t), S_2(t)) = g(t, S_1(t), s_2(t))\}$.

Let

$$\begin{aligned}\mathcal{L}v(t,x_1,x_2) &= rv(t,x_1,x_2) - v_t(t,x_1,x_2) - rx_1v_{x_1}(t,x_1,x_2) - rx_1v_{x_1}(t,x_1,x_2)\\ &\quad -\frac{1}{2}\sigma_1^2x_1^2v_{x_1x_1}(t,x_1,x_2) - \sigma_1\sigma_2x_1x_2v_{x_1}x_2(t,x_1,x_2) - \frac{5}{2}\sigma_2^2x_2^2v_{x_2x_2}(t,x_1,x_2)\end{aligned}$$

Assume $v(t,x_1,s_2)$ is sufficiently smooth so that Itô's rule may be applied to $e^{-rt}v(t,S_1(t),S_2(t))$; this will require at least that v, v_t, v_{x_1}, v_{x_2} exist and are continuous and that the second partials exists and are continuous except possibly along isolated boundaries between regions. Then

$$\begin{aligned}e^{-rt}v(t,S_1(t),S_2(t)) &= v(0,S_1(0),S_2(0)) - \int_0^t e^{-ru}\big[\mathcal{L}v\big](u,s_1(u),s_2(u))\,du\\ &+\int_0^t e^{-ru}\big[v_{x_1}(u,S_1(u),s_2(u))\sigma_1S_1(u) + v_{x_2}(u,S_1(u),s_2(u))\sigma_2S_2(u)\big]\,d\widetilde{W}_1(t)\\ &+\int_0^t e^{-ru}v_{x_2}(u,S_1(u),s_2(u))2\sigma_2S_2(u)\,d\widetilde{W}_2(t) \qquad (1)\end{aligned}$$

Then the linear complimentarity conditions are

$$v(t,x_1,x_2) \geq g(t,x_1,x_2), \quad \text{if } 0\le t\le T,\ x_1,x_2\ge 0; \qquad (2)$$

$$\mathcal{L}v(t,x_1,x_2) \geq 0, \quad \text{for } 0\le t< T, \text{ and } x_1,x_2>0; \qquad (3)$$

$$\mathcal{L}v(t,x_1,x_2) = 0, \quad \text{when } v(t,x_1,x_2) > g(x_1,x_2) \text{ and } 0\le t<T,\ x_1,x_2>0. \qquad (4)$$

This works because, by equation (1), equation (3) implies that $e^{-rt}v(t,S_1(t),S_2(t))$, $0\le t\le T$, is a supermartingale, and equation (4) implies that $e^{-r(t\wedge\tau^*)}v(t\wedge\tau*,S_1(t\wedge\tau^*),S_2(t\wedge\tau^*))$, $0\le t\le T$, is a martingale.

3. For background on this problem, which is a review of multidimensional market modeling as summarized in section 5.4.2 of Shreve, see section 2 of the *Notes to Lectures 9*. For part (b), review the multi-dimensional Girsanov theorem, section 5.4.1 of Shreve. This is also reviewed on Section 4 of the *Notes to Lectures 9.*

(a) Consider a market with three risky assets, whose prices in dollars are $S_1(t)$ and $S_2(t)$ and $Q(t)$.

Let μ_1, μ_2, γ, σ_1, σ_2, and σ_3 be given constants. Write down a model in the form of (5.4.6) in Shreve using a 3-dimensional Brownian motion W ($d=3$) so that the model satisfies the following informally given conditions

$$E[\frac{dS_1(t)}{S_1(t)}\;\Big|\;\mathcal{F}(t)] = \mu_1\,dt \qquad E[\frac{dS_2(t)}{S_2(t)}\;\Big|\;\mathcal{F}(t)] = \mu_2\,dt, \qquad E[\frac{dQ(t)}{Q(t)}\;\Big|\;\mathcal{F}(t)] = \gamma\,dt$$

$$\mathrm{Var}\left(\frac{dS_1(t)}{S_1(t)}\;\Big|\;\mathcal{F}(t)\right)=\sigma_1^2\,dt,\quad \mathrm{Var}\left(\frac{dS_2(t)}{S_2(t)}\;\Big|\;\mathcal{F}(t)\right)=\sigma_2^2\,dt,\quad \mathrm{Var}\left(\frac{dQ(t)}{Q(t)}\;\Big|\;\mathcal{F}(t)\right)=\sigma_3^2\,dt$$

$$\mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)},\frac{dS_2(t)}{S_2(t)}\;\Big|\;\mathcal{F}(t)\right)=\frac{1}{4}\sigma_1\sigma_2\,dt,\quad \mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)},\frac{dQ(t)}{Q(t)}\;\Big|\;\mathcal{F}(t)\right)=\frac{1}{2}\sigma_1\sigma_3\,dt,$$

$$\mathrm{Cov}\left(\frac{dS_2(t)}{S_2(t)},\frac{dQ(t)}{Q(t)}\;\Big|\;\mathcal{F}(t)\right)=\frac{1}{8}\sigma_2\sigma_3\,dt,$$

Hint: In the notation of (5.4.6), start with $\sigma_{ij}=0$ for $j>i$.

Ans: The idea is to look for a system of the form

$$\begin{aligned}
dS_1(t) &= \mu_1 S_1(t)\,dt+\sigma_1 S_1(t)\,dW_1(t)\\
dS_2(t) &= \mu_2 S_2(t)\,dt+\sigma_2 S_2(t)\left[\rho\,dW_1(t)+\sqrt{1-\rho^2}\,dW_2(t)\right]\\
dQ(t) &= \gamma Q(t)\,dt+\sigma_3 Q(t)\left[\eta_1\,dW_1(t)+\eta_2\,dW_2(t)+\sqrt{1-\eta_1^2-\eta_2^2}\,dW_3(t)\right],
\end{aligned}$$

where $|\rho|\le 1$ and $\eta_1^2+\eta_2^2+\eta_3^2=1$. In this form, all the requirements about me**Ans** and variances are automatically satisfied. It remains to choose the other parameters to fit the prescribed covariances. Since $[dS_1(t)/S_1(t)][dS_2(t)/S_2(t)]=\rho\sigma_1\sigma_2$, we want,

$$\frac{\sigma_1\sigma_2}{4}=\mathrm{Cov}(dS_1(t)/S_1(t),dS_2(t)/S_2(t)\;\Big|\;\mathcal{F}(t))=\rho\sigma_1\sigma_2.$$

Thus $\rho=1/4$.

Since

$$\frac{1}{2}\sigma_1\sigma_3=\mathrm{Cov}(dS_1(t)/S_1(t),dQ(t)/Q(t)\;\Big|\;\mathcal{F}(t))=\eta_1\sigma_1\sigma_3,$$

we find $\eta_1=1/2$. Finally,

$$\frac{1}{8}\sigma_2\sigma_3=\frac{dS_2(t)}{S_2(t)}\frac{dQ(t)}{Q(t)}=\sigma_2\sigma_3\left[\rho\eta_1+\eta_2\sqrt{1-\rho^2}\right]\,dt=\sigma_2\sigma_3\left[\frac{1}{8}+\eta_2\sqrt{\frac{15}{16}}\right]$$

Thus, $\eta_2=0$.

In summary,

$$\begin{aligned}
dS_1(t) &= \mu_1 S_1(t)\,dt+\sigma_1 S_1(t)\,dW_1(t)\\
dS_2(t) &= \mu_2 S_2(t)\,dt+\sigma_2 S_2(t)\left[\frac{1}{4}\,dW_1(t)+\frac{\sqrt{15}}{4}\,dW_2(t)\right]\\
dQ(t) &= \gamma Q(t)\,dt+\sigma_3 Q(t)\left[\frac{1}{2}\,dW_1(t)+\frac{\sqrt{3}}{2}\,dW_3(t)\right],
\end{aligned}$$

(b) Assume that σ_1, σ_2 and σ_3 are stricly positive. Assume there is a unique risk-neutral measure $\widetilde{\mathbf{P}}$ where the risk free rate is $R(t)$, an adapted process. Find $\Theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t))$ such that $\widetilde{W}(t) = W(t) + \int_0^t (\theta_1(u), \theta_2(u), \theta_3(u))\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}$.

(See page 228 for a review of the background to this problem.) Let $R(t)$ be the short rate. The risk-neutral measure is defined by

$$\widetilde{\mathbf{P}}(A) = E[\mathbf{1}_A \exp\{-\int_0^T \Theta(u)\, dW(u) - \frac{1}{2}\int_0^T \|\Theta(u)\|^2\, du\}]$$

where $\Theta(u) = (\theta_1(u), \theta_2(u), \theta_3(u))$ satisfies the market price of risk equations:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ \frac{\sigma_2}{4} & \sigma_2 \frac{\sqrt{15}}{4} & 0 \\ \frac{\sigma_3}{2} & 0 & \sigma_3 \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} \theta_1(u) \\ \theta_2(u) \\ \theta_3(u) \end{pmatrix} = \begin{pmatrix} \mu_1 - R(t) \\ \mu_2 - R(t) \\ \gamma - R(t) \end{pmatrix}.$$

Thus

$$\begin{aligned} \theta_1(u) &= \frac{\mu_1 - R(t)}{\sigma_1}, \\ \theta_2(u) &= \frac{4}{\sigma_2\sqrt{15}}\left(\mu_2 - R(t) - \frac{\sigma_2}{4}\frac{\mu_1(t) - R(t)}{\sigma_1}\right) \\ \theta_3(u) &= \frac{2}{\sigma_3\sqrt{3}}\left(\gamma - R(t) - \frac{\sigma_3}{2}\frac{\mu_1 - R(t)}{\sigma_1}\right) \end{aligned}$$

Under the risk-neutral measure,

$$\widetilde{W_1}(t) = (W_1(t) + \int_0^t \theta_1(u)\, du, W_2(t) + \int_0^t \theta_2(u)\, du, W_3(t) + \int_0^t \theta_3(u)\, du)$$

is a Brownian motion under $\widetilde{\mathbf{P}}$.

4. Shreve, Exercise 5.8. This exercise should help you understand Theorem 9.2.1 on page 377 when you get to it.

Ans: (i) $D(t)V(t)$ is a martingale and so by the martingale representation theorem there is an $\{\mathcal{F}(t);\, t \geq 0\}$-adapted process $\tilde{\Gamma}(t)$ such that

$$D(t)V(t) = D(0)V(0) + \int_0^t \tilde{\Gamma}(u)\, d\widetilde{W}(u) = V(0) + \int_0^t \tilde{\Gamma}(u)\, d\widetilde{W}(u).$$

By calculating $d[D(t)V(t)]$ using $d[D(t)] = -R(t)D(t)\, dt$,

$$-R(t)D(t)V(t)\, dt + D(t)\, dV(t) = d[D(t)V(t)] = \tilde{\Gamma}(t)\, d\widetilde{W}(t).$$

By solving for $dV(t)$,

$$dV(t) = R(t)V(t)\,dt + \frac{\tilde{\Gamma}(t)}{D(t)}\,d\widetilde{W}(t).$$

(ii) It is a general fact that if X is a random variable for which $\mathbf{P}(X > 0) = 1$ and $\mathcal{G}$ is a σ-algebra, $E[X \mid \mathcal{G}] > 0$ with probability one also. To see why this must be true, first observe that if $\mathbf{P}(A) > 0$ it must be true that $E[X\mathbf{1}_A] > 0$ if $\mathbf{P}(X > 0) = 1$; thus, if for some event A, $E[X\mathbf{1}_A] = 0$ it must be true that $\mathbf{P}(A) = 0$. Now let A be the set $\{\omega; E[X \mid \mathcal{G}](\omega) = 0\}$. We want to show $\mathbf{P}(A) = 0$. Since A is an $\mathcal{G}$-measurable event, because it is defined as the set where the $\mathcal{G}$-measurable random variable $E[X \mid \mathcal{G}]$ equals zero, $E[\mathbf{1}_A X] = E[\mathbf{1}_A E[X \mid \mathcal{G}]]$. But this expectation is zero because $E[X \mid \mathcal{G}] = 0$ on A by definition. Thus $E[\mathbf{1}_A X] = 0$ and hence $\mathbf{P}(A) = 0$.

Applying the principle we just proved to $V(t) = \tilde{E}[D(t)V(T) \mid \mathcal{F}(t)]$, if $\widetilde{\mathbf{P}}(V(T) > 0) = 1$, then $\widetilde{\mathbf{P}}(V(t) > 0) = 1$ also.

(iii) Because of the result of (ii), it is possible to define $\sigma(t) = \tilde{\Gamma}(t)/[D(t)V(t)]$ for all $t \leq T$. By substituting this into the stochastic differential of $dV(t)$ obtained in part (i),

$$dV(t) = R(t)V(t)\,dt + \sigma(t)V(t)\,d\widetilde{W}(t).$$