Homework 7 (Due 3/30/2016)

Math 622

March 24, 2016

1. Theorem 8.4.2 of the *Chapter 8* states the linear complementarity conditions for solving the perpetual put when the price follows the Black-Scholes, constant volatility model. The purpose of this problem is to develop and prove the linear complementarity and side conditions for the problem of finding

$$v(x) = \sup \left\{ E[e^{-r\tau} g(S_\tau \middle| S(0){=}x]];\ \tau \text{ is a stopping time} \right\}$$

where $x \geq 0$, g is a bounded, positive function, and

$$dS(t) = b(S(t))\,dt + \sigma(S(t))S(t)\,dW(t).$$

Here $b(x) \geq 0$ if $x \geq 0$ so that $S(t)$ always remains positive. The coefficient $b(x)$ is not necessarily rx. We are considering the optimization problem in a more general context.

a) Show that Theorem 8.4.1 of *Chapter 8* remains true when $(K-x)^+$ is replaced by $g(x)$, and $v(x)$ is defined as above.

b) Show that Theorem 8.4.2 of the *Chapter 8* is also true when $(K-x)^+$ is replaced by $g(x)$ and the left-hand sides of equation (14) and (15) in Theorem 4.2 are replaced by $ru(x) - b(x)u_x(x) - \frac{1}{2}\sigma^2(x)x^2u_{xx}(x)$. (Follow the steps of the proof in the lecture notes.)

2. This problem illustrates another extension of the optimal stopping method of Theorems 8.4.1 and 8.4.2. Let $a \geq 0$. Let $X(0) > 0$, let W be a Brownian motion. and let ρ be the first time $X(0) + at + \sigma W(t) = 0$. Define

$$X(t) = X(0) + a(t \wedge \rho) + \sigma W(t \wedge \rho),$$

Thus, once X hits 0, it stays there. Let $\{\mathcal{F}(t);\, t \geq 0\}$ be the filtration generated by W. Let $K > 0$ and $r > 0$, and consider the following problem of finding

$$v(x) = \max\{E\Big[e^{-r\tau}(K - X(\tau))^+ \;\Big|\; X(0) = x\Big];\, \tau \text{ is a stopping time.}\}$$

and an optimal exercise time τ^*, that is, a stopping time satisfying

$$v(x) = E\Big[e^{-r\tau}(K - X(\tau^*))^+ \;\Big|\; X(0) = x\Big], \qquad x \geq 0.$$

(We can think of this as a perpetual put problem. However this price model is not generally used in finance (although it does allow bankruptcy), and it is not a risk neutral model. Nevertheless, it makes for a simple, hands-on problem of optimal stopping.)

(a) Show that $v(x) \leq K$ for all $x \geq 0$, that $v(0) = K$ and, that it never makes sense to exercise after time ρ.

(b) Following the lecture notes, show the following: Suppose $u(x)$, $x \geq 0$, satisfies

(a) $u(x) \geq (K - x)^+$ for all $x \geq 0$ and $u(0) = K$;

(b) u is bounded;

(c) $e^{-r(t\wedge\rho)}u(X(t \wedge \rho))$ is a supermartingale;

(d) if τ^* is the first time $X(t)$ hits the region $\mathcal{S} = \{x;\, u(x) = (K - x)^+\}$, $e^{-r(t\wedge\tau^*)}u(X(t \wedge \tau^*))$ is a martingale.

Then $u(x) = v(x)$ and τ^* is an optimal exercise time.

(c) Following the lecture notes, show that if $u(x)$, $x \geq 0$ is a function such that $u(x)$ and $u'(x)$ are continuous and $u''(x)$ exists and is continuous except possibly at a finite number of points where it has jump discontinuities, and if

$$u(x) \geq (K - x)^+ \text{ for all } x \geq 0,\ u(0) = K, \text{ and } u \text{ is bounded} \tag{1}$$

$$-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) \geq 0, \quad \text{if } x > 0. \tag{2}$$

$$-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) = 0, \quad \text{if } u(x) > (K - x)^+. \tag{3}$$

then u satisfies the conditions (a)-(d) in part (a) and hence $u(x) = v(x)$.

(d) The general solution to

$$\frac{1}{2}\sigma^2 v''(x) + av'(x) - rv(x) = 0, \qquad x > L^* \tag{4}$$

on any interval has the form $Ae^{-\beta_1 x} + Be^{-\beta_2 x}$ for certain constants β_1 and β_2. Find these constants by substituting $e^{-\beta x}$ into (4) and finding a quadratic equation that β must solve for $e^{-\beta x}$ to be a solution.

(e) Suppose that

$$\frac{K(a + \sqrt{a^2 + 2\sigma^2 r})}{\sigma^2} > 1. \tag{5}$$

Find a solution v to the system of equations (1), (2), (3), following the method used in text or class notes to solve the perpetual put equations for the Black-Scholes price. (We call it v now instead of u because we know it will be the value function). Start from the guess that the exercise (stopping) region has the form $\mathcal{S} = \{x : v(x) = (K-x)^+\} = [0, L^*]$, for some constant $L^* \geq 0$. Use part (d) to represent the solution to v on the continuation region and find L^* by requiring that v and v' be continuous. You should find $L^* = K - \frac{\sigma^2}{a+\sqrt{a^2+2\sigma^2 r}}$. Show that the solution found in (ii) satisfies conditions (1) and (2).

3. Let $v(x) = \sup\{\tilde{E}[e^{-r\tau}(K - S(\tau)) + \big| S(0){=}x]; \tau \text{ is a stopping time.}\}$, where

$$dS(t) = (r-a)S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t).$$

Here $a > 0$. This is the problem of pricing an perpetual put when the asset pays dividends at rate a.

This problem is posed as Exercise 8.5 in Shreve. You may solve this by following the method outlined in Exercise 8.5; for that you will have to read Shreve, pages 346-352. Or you can follow the strategy of the class notes as follows.

a) Use problem 1 to derive the linear complementarity equations (Note that the only difference from the case with no dividends is the appearance of $(r-a)x$ instead of rx in the partial differential operator.)

b) Find the general solution to $rv(x) - (r-a)xv'(x) - (\sigma^2/2)x^2 v''(x) = 0$. You can do this as in problem 8.3: try a solution of the from x^p and find an equation for p. You should find that the solution that is bounded has the form $Ax^{-\gamma}$ where

$$\gamma = \frac{(r - a - \frac{1}{2}\sigma^2) + \sqrt{(r - a - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$

This is the same γ that is defined in Exercise 8.5.

c) Assume that the stopping region is of the form $[0, L^*]$, as we did for the case of no dividends. Use smooth fit to find $v(x)$. (Express your answer in terms of γ.)