Homework 7 (Sol)

Math 622

April 11, 2016

1. Theorem 8.4.2 of the *Chapter 8* states the linear complementarity conditions for solving the perpetual put when the price follows the Black-Scholes, constant volatility model. The purpose of this problem is to develop and prove the linear complementarity and side conditions for the problem of finding

$$v(x) = \sup \left\{ E[e^{-r\tau} g(S_\tau \Big| S(0)=x]];\ \tau \text{ is a stopping time} \right\}$$

where $x \geq 0$, g is a bounded, positive function, and

$$dS(t) = b(S(t))\,dt + \sigma(S(t))S(t)\,dW(t).$$

Here $b(x) \geq 0$ if $x \geq 0$ so that $S(t)$ always remains positive. The coefficient $b(x)$ is not necessarily rx. We are considering the optimization problem in a more general context.

a) Show that Theorem 8.4.1 of the *Chapter 8* remains true when $(K-x)^+$ is replaced by $g(x)$, and $v(x)$ is defined as above.

Ans: The steps of Theorem 8.4.1 of the *Chapter 8* follow through without significant change because there is no place that the particular stochastic differential equation model for $S(t)$ is used, and only the boundedness of the payoff function matters in the other steps.

b) Show that Theorem 8.4.2 of the *Notes to Lecture 8* is also true when $(K-x)^+$ is replaced by $g(x)$ and the left-hand sides of equation (8.14) and (8.15) in Theorem 8.4.2 are replaced by $ru(x) - b(x)u_x(x) - \frac{1}{2}\sigma^2(x)x^2u_{xx}(x)$. (Follow the steps of the proof in the lecture notes.)

Ans: (b) We will show: suppose

(i) $u(x) \geq g(x)$ for $x \geq 0$;

(ii) u is bounded;

(iii) u, u_x are continuous and u_{xx} is piecewise continuous and

$$ru(x) - b(x)u_x(x) - \frac{1}{2}\sigma^2(x)x^2u_{xx}(x) \geq 0, \quad \text{x¿0;}$$

(iv) On the set $\{x;\, u(x) > g(x)\}$,

$$ru(x) - b(x)u_x(x) - \frac{1}{2}\sigma^2(x)x^2u_{xx}(x) = 0,$$

then,

$$u(x) = \sup\{E[e^{-r\tau}g(S(T))|S(0) = x];\, \tau \text{ is a stopping time}\}$$

and $\tau^* = \inf\{t : u(S(t)) = g(S(t))\}$ is an optimal exercise time.

Indeed, by Itô's rule,

$$\begin{aligned} e^{-rt}u(S(t)) &= u(S(0)) \\ &+ \int_0^t e^{-rs}\Big[-ru(S(s)) + rb(S(s))u_x(S(s)) + \frac{1}{2}\sigma^2(S(s))s^2(s)u_{xx}(S(s))\Big]\,ds \\ &\quad + \int_0^t e^{-rs}u_x(S(s))\sigma(S(s))S(s)\,dW(s) \end{aligned}$$

The last term is a martingale and the middle term is a decreasing function of t with probability 1, because of (iii). Thus $e^{-rt}u(S(t))$ is a super-martingale, and this is precisely what the extension of Theorem 4.1 derived in part (a) requires.

Let $\tau*$ be defined as above. Then, for $t < \tau*$, the integrand of the ordinary integral previous formula is equal to 0, because of (iv). Thus

$$\begin{aligned} e^{-r(t\wedge\tau*)}u(S(t\wedge\tau*)) &= u(S(0)) \\ &+ \int_0^{t\wedge\tau*} e^{-rs}u_x(S(s))\sigma(S(s))S(s)\,dW(s) \end{aligned}$$

But since the right-hand side is now just a martingale, we find that $e^{-r(t\wedge\tau*)}u(S(t\wedge\tau*))$ is a martingale. This is just what Theorem 4.1 requires. Hence it follow that $u(x) = v(x)$ and that $\tau*$ is an optimal exercise time.

2. This problem illustrates another extension of the optimal stopping method of Theorems 8.4.1 and 8.4.2. Let $a \geq 0$. Let $X(0) > 0$, let W be a Brownian motion. and let ρ be the first time $X(0) + at + \sigma W(t) = 0$. Define

$$X(t) = X(0) + a(t\wedge\rho) + \sigma W(t\wedge\rho),$$

Thus, once X hits 0, it stays there. Let $\{\mathcal{F}(t);\, t \geq 0\}$ be the filtration generated by W. Let $K > 0$ and $r > 0$, and consider the following problem of finding

$$v(x) = \max\{E\left[e^{-r\tau}(K - X(\tau))^+ \;\middle|\; X(0) = x\right];\, \tau \text{ is a stopping time.}\}$$

and an optimal exercise time τ^*, that is, a stopping time satisfying

$$v(x) = E\left[e^{-r\tau}(K - X(\tau^*))^+ \;\middle|\; X(0) = x\right], \qquad x \geq 0.$$

(We can think of this as a perpetual put problem. However this price model is not generally used in finance (although it does allow bankruptcy), and it is not a risk neutral model. Nevertheless, it makes for a simple, hands-on problem of optimal stopping.)

(a) Show that $v(x) \leq K$ for all $x \geq 0$, that $v(0) = K$ and, that it never makes sense to exercise after time ρ.

Ans: Suppose you have not exercised until ρ, where ρ is the first time $X(t)$ hits 0. Then exercising at ρ give you a discounted return of $e^{-r\rho}K$. If you wait an additional h units of time to exercise, your payoff is $e^{-r(\rho+h)}K < e^{-r\rho}K$. Hence, to maximize expected return you should never exercise the option after time ρ.

(b) Following the text or lecture, show the following: Suppose $u(x)$, $x \geq 0$, satisfies

(a) $u(x) \geq (K - x)^+$ for all $x \geq 0$ and $u(0) = K$;

(b) u is bounded;

(c) $e^{-r(t\wedge\rho)}u(X(t \wedge \rho))$ is a supermartingale;

(d) if τ^* is the first time $X(t)$ hits the region $\mathcal{S} = \{x;\, u(x) = (K - x)^+\}$, $e^{-r(t\wedge\tau^*)}u(X(t \wedge \tau^*))$ is a martingale.

Then $u(x) = v(x)$ and τ^* is an optimal exercise time.

Ans: The argument is the same as in the lecture notes. One needs to be careful about stopping times. Notice that condition (c) of the problem statement says that $e^{-r(\rho\wedge t)}u(X(\rho \wedge t))$ is a supermartingale, where ρ is the first time $X(t)$ hits zero. But we know from part a) that we never exercise after ρ. Therefore, in proving that u is the optimal value function it is only necessary to consider stopping times τ satisfying $\tau \leq \rho$.

For stopping times $\tau \le \rho$, condition (c) implies $e^{-r(\tau\wedge t)}u(X(\tau \wedge t))$ will be a supermartingale. Mathematically this implies,

$$u(x) \ge E[e^{-r\tau\wedge\rho}u(X(t\wedge\tau)) \;\Big|\; X(0) = x] \ge E[e^{-r(t\wedge\tau)}(K - X(t\wedge\tau))^+ \;\Big|\; X(0) = x].$$

By boundedness of $(K-x)^+$, we can interchange limits as $t \to \infty$ and expectation, so, since $\lim_{t\to\infty} e^{-r(\tau\wedge t)}(K - X(t\wedge\tau))^+ = e^{-r\tau}(K - X(\tau))^+$,

$$u(x) \ge E[e^{-r\tau}(K - X(\tau))^+ \;\Big|\; X(0) = x],$$

and hence, since this is true for any stopping time with $\tau \le \rho$,

$$u(x) \ge \max_{\tau\in\mathcal{T}} E[e^{-r(\tau)}(K - X(\tau))^+ \;\Big|\; X(0) = x] = v(x).$$

where $\mathcal{T}$ is the set of stopping times τ with $\tau \le \rho$.

On the other hand, by (d),

$$u(x) = E[e^{-r(\tau^*\wedge t)}u(X(\tau^*\wedge t)) \;\big|\; X(0) = x]$$

Now $u(X(\tau^*\wedge t)) = (K - X(\tau^*))^+$ if $t > \tau^*$, and if $\tau^* = \infty$, $\lim_{t\to\infty} e^{-r(\tau^*\wedge t)}u(X(\tau^* \wedge t)) = \lim_{t\to\infty} e^{-rt)}u(X(t)) = 0$, which is the payoff at ∞. It follows by taking $t \to \infty$ that $u(x) = E[e^{-r\tau^*}(K - X(\tau^*)^+ \;\big|\; X(0) = x]$. Thus,

$$u(x) \le \max_{\tau\in\mathcal{T}} E[e^{-r\tau}(K - X(\tau))^+ \;\Big|\; X(0) = x] = v(x).$$

We have shown that $u(x) \ge v(x)$ and $u(x) \le v(x)$, and hence

$$u(x) = \max_{\tau\in\mathcal{T}} E[e^{-r\tau}(K - X(\tau))^+ \;\Big|\; X(0) = x] = v(x),$$

and τ^* is an optimal stopping time.

(c) Following the text or lecture, show that if $u(x)$, $x \ge 0$ is a function such that $u(x)$ and $u'(x)$ are continuous and $u''(x)$ exists and is continuous except possibly at a finite number of points where it has jump discontinuities, and if

$$u(x) \ge (K-x)^+ \text{ for all } x \ge 0,\ u(0) = K, \text{ and } u \text{ is bounded} \tag{1}$$

$$-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) \ge 0, \quad \text{if } x > 0. \tag{2}$$

$$-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) = 0, \quad \text{if } u(x) > (K-x)^+. \tag{3}$$

then u satisfies the conditions (a)-(d) in part (a) and hence $u(x) = v(x)$.

Ans: We have only to check the conditions (c) and (d), as (a) and (b) are assumed. By Itô's rule,

$$e^{-rt}u(X(t)) = u(x) - \int_0^{t\wedge\rho} e^{-ru}\left(ru(X(u)) - au'(X(s)) - \frac{1}{2}u''(X(u))\right) du + \int_0^{t\wedge\rho} e^{-ru}u'(X(u))\sigma\, dW(u).$$

By equations (2) and (3) of the problem statement, the integrand of the ordinary integral is non-positive and hence the integral is a non-increasing function of t. Since the stochastic integral term is a martingale, it follows that $e^{-rt}u(X(t))$ is a super-martingale.

Moreover, by equation (3), the integrand of the 'dt' is 0 as long as $X(t)$ is in the region where $u(x) > (K-x)^+$, that is, as long as $t < \tau^*$. We have argued above that we will always exercise at or before time ρ. Thus $\tau* \le \rho$ with probability one. It follows from the above that

$$\begin{aligned} e^{-r(\tau^*\wedge t)}u(X(\tau^*\wedge t)) &= u(x) - \int_0^{\tau^*\wedge t} e^{-ru}\left(ru(X(u)) - au'(X(s)) - \frac{1}{2}u''(X(u))\right) du \\ &\quad + \int_0^{\tau^*\wedge t} e^{-ru}u'(X(u))\sigma\, dW(u) \\ &= u(x) + \int_0^{\tau^*\wedge t} e^{-ru}u'(X(u))\sigma\, dW(u). \end{aligned}$$

This is a martingale. Hence we have verified conditions (a)-(d), and thus $u(x) = v(x)$.

(d) The general solution to

$$\frac{1}{2}\sigma^2 v''(x) + av'(x) - rv(x) = 0, \qquad x > L^* \tag{4}$$

on any interval has the form $Ae^{-\beta_1 x} + Be^{-\beta_2 x}$ for certain constants β_1 and β_2. Find these constants by substituting $e^{-\beta x}$ into (4) and finding a quadratic equation that β must solve for $e^{-\beta x}$ to be a solution.

Ans: By substituting $e^{-\beta x}$ in $\frac{1}{2}\sigma^2 u''(x) + au'(x) - ru(x) = 0$, we obtain,

$$(\frac{1}{2}\sigma^2\beta^2 - a\beta - r)e^{-\beta x} = 0.$$

Thus $e^{-\beta x}$ solves equation (3), if

$$\beta_1 = \frac{a + \sqrt{a^2 + 2r\sigma^2}}{\sigma^2} \quad \text{or} \quad \beta_2 = \frac{a - \sqrt{a^2 + 2r\sigma^2}}{\sigma^2}.$$

Observe that $\beta_2 < 0 < \beta_1$.

(e) Suppose that

$$\frac{K(a+\sqrt{a^2+2\sigma^2 r})}{\sigma^2} > 1. \tag{5}$$

Find a solution v to the system of equations (1), (2), (3), following the method used in text or class notes to solve the perpetual put equations for the Black-Scholes price. (We call it v now instead of u because we know it will be the value function). Start from the guess that the exercise (stopping) region has the form $\mathcal{S} = \{x : v(x) = (K-x)^+\} = [0, L^*]$, for some constant $L^* \geq 0$. Use part (d) to represent the solution to v on the continuation region and find L^* by requiring that v and v' be continuous. You should find $L^* = K - \frac{\sigma^2}{a+\sqrt{a^2+2\sigma^2 r}}$. Show that the solution found in (ii) satisfies conditions (1) and (2).

Ans: From part (d), we see that the general solution to equation (3) on the interval $x > L^*$ is thus $Ae^{-\beta_1 x} + Be^{-\beta_2 x}$. This is bounded on $L^* < x < \infty$ only if $B = 0$. So if $v(x)$ has the form we want, it should satisfy

$$v(s) = \begin{cases} K-x, & \text{if } 0 \leq x \leq L_*; \\ Ae^{-\beta_1 x}, & \text{if } L^* < x. \end{cases}$$

In order that v be continuously differentiable, the values of $v(x)$ and $v'(x)$ must match as x approaches L^* from the left and the right. This requires,

$$K - L^* = Ae^{-\beta_1 L^*}, \qquad \text{and} \qquad -1 = -\beta_1 Ae^{-\beta_1 L^*}.$$

These are the smooth fit equations. Solving them yields

$$L^* = K - \frac{1}{\beta_1} = K - \frac{\sigma^2}{a+\sqrt{a^2+2r\sigma^2}}.$$

By the assumption made in equation (5) of the problem statement, this value of L^* is positive. Also, it is clearly less than K. From this expression for L^*,

$$A = \frac{e^{\beta_1 L^*}}{\beta_1} = \frac{e^{\beta_1 K - 1}}{\beta_1}.$$

By construction $-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) = 0$ for $x > L^*$. For $0 \leq x \leq L^*$, $u(x) = K - x$ and so $-\frac{1}{2}\sigma^2 u''(x) - au'(x) + ru(x) = a + r(K-x) \geq 0$. Thus equation (2) of the problem statement is satisfied:

$$\frac{1}{2}\sigma^2 u''(x) + au'(x) - ru(x) \geq 0 \quad \text{for all } x > 0.$$

To prove $v(x) \geq (K-x)^+$ for all x, it suffices to prove that $v(x) > (K-x)^+ > 0$ for all $x > L^*$. This requires proving both $v(x) > 0$ and $h(x) = v(x) - (K-x) > 0$ for $x > L^*$. Clearly $v(x) = Ae^{-\beta_1 x} > 0$ for $x > L^*$. As for h, observe that $h(L^*) = 0$ and for $x > L^*$, $h'(x) = -A\beta_1 e^{-\beta_1 x} + 1$. Since $-A\beta_1 e^{-\beta_1 L^*} = -1$ by smooth fit, it follows that $h'(x) = -e^{-\beta_1(x-L)} + 1 > 0$ for $x > L^*$. Thus h is strictly increasing on the interval $x > L^*$, and so $h(x) > h(L^*) = 0$ for $x > L^*$.

Additional Remark. What happens if $K\beta_1 = \frac{K(a+\sqrt{a^2+2r\sigma^2})}{\sigma^2} \leq 1$? We claim that then

$$v(x) = Ke^{-\beta_1 x} > (K-x)^+ \quad \text{for all } x > 0.$$

Thus v solves equations (1), (2), (3) of the problem statement and $\{x; v(x) = (K-x)^+\} = \{0\}$. This means that $L^* = 0$ and it is optimal to stop only when $X(t)$ hits 0. To prove that $v(x) = Ke^{-\beta_1 x} > (K-x)^+$ when $x > 0$, let $h(x) = Ke^{-\beta_1 x} - (K-x)$. Observe that

$$v'(x) = -K\beta_1 e^{-\beta_1 x} + 1 > 0, \qquad \text{for all } x > 0.$$

if $K\beta_1 \leq 1$. Since $h(0) = 0$, it follows that $h(x) > 0$, or equivalently, $v(x) > (K-x)$, for $x \geq 0$. Also, $v(x) > 0$ for $x \geq 0$ and so $v(x) > (K-x)^+$ for $x > 0$.

3. Let $v(x) = \sup\{\tilde{E}[e^{-r\tau}(K - S(\tau)) + \big| S(0)=x]; \tau \text{ is a stopping time.}\}$, where

$$dS(t) = (r-a)S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t).$$

Here $a > 0$. This is the problem of pricing an perpetual put when the asset pays dividends at rate a.

This problem is posed as Exercise 8.5 in Shreve. You may solve this by following the method outlined in Exercise 8.5; for that you will have to read Shreve, pages 346-352. Or you can follow the strategy of the class notes as follows.

a) Use problem 1 to derive the linear complementarity equations (the equations (Note that the only difference from the case with no dividends is the appearance of $(r-a)x$ instead of rx in the partial differential operator.)

Ans: We seek a bounded function v such that v and v' are continuous and v'' is piecewise continuous, such that $v(x) \geq (K-x)^+$, and such that

$$rv(x) - (r-a)xv'(x) - \frac{\sigma^2}{2}x^2 v''(x) \geq 0, \quad \text{for all } x > 0; \tag{6}$$

and

$$rv(x) - (r-a)xv'(x) - \frac{\sigma^2}{2}x^2 v''(x) = 0, \quad \text{when } v(x) > (K-x)^+. \tag{7}$$

b) Find the general solution to $rv(x) - (r-a)xv'(x) - (\sigma^2/2)x^2v''(x) = 0$. You can do this as in problem 8.3: try a solution of the from x^p and find an equation for p. You should find that the solution that is bounded has the form $Ax^{-\gamma}$ where

$$\gamma = \frac{(r-a-\frac{1}{2}\sigma^2) + \sqrt{(r-a-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$

This is the same γ that is defined in Exercise 8.5.

Ans: We shall look for a solution such that $v(x) = (K-x)$ for $0 \le x \le L^*$, where $0 < L^* < K$, and $v(x) > (K-x)^+$ for $x > L^*$. Therefore for $x > L^*$, v must be a bounded solution to equation (7). By plugging in x^{-q} into this equation, we see that it will solve (7) if

$$r + (r-a)q - \frac{\sigma^2}{2}q(q+1) = 0. \tag{8}$$

Thus, the general solution to (7), is

$$v(x) = Ax^{-\gamma} + Bx^{-\kappa},$$

where

$$\begin{aligned} \gamma &= \frac{(r-a-\frac{1}{2}\sigma^2) + \sqrt{(r-a-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}; \\ \kappa &= \frac{(r-a-\frac{1}{2}\sigma^2) - \sqrt{(r-a-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2} \end{aligned}$$

Clearly, $\gamma > 0$ and $\kappa < 0$, and therefore $Bx^{-\kappa}$ is unbounded on any interval of the form (L, ∞). Therefore, if we are to have a bounded solution of the form we are seeking, it must be true that $v(x) = Ax^{-\gamma}$ for $x > L^*$.

c) Assume that the stopping region is of the form $[0, L^*]$, as we did for the case of no dividends. Use smooth fit to find $v(x)$. (Express your answer in terms of γ.)

Ans: Now we will impose the smooth fit condition. In order that v be continuous at the boundary $x = L^*$, it is necessary that

$$K - L^* = A(L^*)^{-\gamma}.$$

In order that v' be continuous at L^*, it is necessary that

$$-1 = -\gamma A(L^*)^{-\gamma}(L^*)^{-1}.$$

By using the first equation in the second,

$$-1 = -\frac{\gamma}{L^*}(K - L^*).$$

Thus $L^* = \dfrac{\gamma}{1+\gamma}$, and $A = \gamma(L^*)^{\gamma+1}$.

It remains to check for the solution we have constructed that equations (6) and (7) are satisfied. Since $rv(x) - (r-a)xv'(x) - \frac{\sigma^2}{2}x^2v''(x) = 0$ by construction for $x > L^*$, we only need to check (6) for $x < L*$, where $v(x) = K - x$. In this case, $v'(x) = -1$ and $v''(x) = 0$. for $x < L^*$, and

$$\begin{aligned} rv(x) - (r-a)xv'(x) - \frac{\sigma^2}{2}x^2v''(x) &= r(K-x) + (r-a)x = rK - ax \\ \geq rK - aL^* = K\frac{r(1+\gamma) - a\gamma}{1+\gamma} \end{aligned}$$

Notice that equation (8) implies

$$r(1+\gamma) - a\gamma = r + (r-a)\gamma = \frac{\sigma^2}{2}\gamma(\gamma+1) > 0.$$

Thus $rv(x) - (r-a)xv'(x) - \dfrac{\sigma^2}{2}x^2v''(x) > 0$ for $x < L^*$, as we needed.

Finally, it is necessary to show that if $x > L^*$, $v(x) = Ax^{-\gamma} > (K-x)^+$ for $x > L^*$. Since, clearly, $v(x) > 0$, it suffices to show that $v(x) > K - x$ for $x > L^*$. Let $f(x) = v(x) - (K-x) = Ax^{-\gamma} - (K-x)$. By the choice of A and L^* to match v and v' at the boundary, $f(L^*) = 0$, and $f'(L^*) = 0$. Now $f'(x) = 1 - \gamma Ax^{-\gamma-1}$ and this is a strictly increasing function because $\gamma > 0$. Thus, since $f'(L^*) = 0$, we have $f'(x) > 0$ for all $x > L^*$. But this implies that $f(x)$ is strictly increasing for $x > L^*$, and hence $0 < f(x) = v(x) - (K-x)$ for $x > L^*$.