Homework 6 (Due 3/23/2016)

Math 622

March 10, 2016

1. Assume that S satisfies the risk-neutral, Black-Scholes model,

$$dS_t = rS_t\,dt + \sigma S_t\,\mathrm{d}\widetilde{W}(t).$$

This problem is about analyzing the the average strike call with payoff

$$V_T = (S_T - \frac{1}{T}\int_0^T S_u du)^+.$$

Let $Y_t = \int_0^t S_u\,\mathrm{d}u$.

a) Show that the price of the can be written as $V(t) = v(t, S_t, Y_t)$ and give an expression for $v(t,x,y)$ of the form $v(t,x,y) = \tilde{E}[L(t,x,y)]$ where L is a random variable, and give an explicit formula for $L(t,x,y)$ in terms of t, x, y and the process $\{W(u) - W(t);\ u \geq t\}$.

b) Find a p.d.e. for v. Be sure to specify the domain of (t,x,y)-space where the equation holds. Derive or write down all boundary and terminal conditions. You will have to specify a boundary condition that fixes the value of v, either as $y \to \infty$ or $y \to -\infty$.

2. Extra Credit (5 points) Let $C(t)$ be the price of a European call with strike K expiring at time T. Let $A(t)$ be the price of the corresponding American call. No particular model is put on the price process, except that we assume no dividends are paid on the asset. Use a no-arbitrage argument to show that $C(t) > (S_t - K)^+$ for all $t < T$, as long as S_T can fall to either side of K with positive probability. Use this result to show that $A(t) = C(t)$ for all t and that early exercise is never optimal. (It is also helpful to keep this observation in mind: If $A(t) > (S_t - K)^+$ then it is not optimal to exercise the American option at time t since trading the option itself gives higher pay off than exercising the option).