Homework 6 (Due 3/23/2016)

Math 622

March 24, 2016

1. Assume that S satisfies the risk-neutral, Black-Scholes model,

$$dS_t = rS_t\,dt + \sigma S_t\,\mathrm{d}\widetilde{W}(t).$$

This problem is about analyzing the the average strike call with payoff

$$V_T = (S_T - \frac{1}{T}\int_0^T S_u du)^+.$$

Let $Y_t = \int_0^t S_u\,\mathrm{d}u$.

a) Show that the price of the can be written as $V(t) = v(t, S_t, Y_t)$ and give an expression for $v(t,x,y)$ of the form $v(t,x,y) = \tilde{E}[L(t,x,y)]$ where L is a random variable, and give an explicit formula for $L(t,x,y)$ in terms of t, x, y and the process $\{W(u) - W(t);\, u \geq t\}$.

Ans: By the Markov property of $(S(t), Y(t))$

$$V(t) = e^{-r(T-t)}\tilde{E}[(S(T) - (Y(T)/T))^+ \big| \mathcal{F}(t)] = v(t, S(t), Y(t)),$$

where

$$v(t,x,y) = e^{-r(T-t)}\tilde{E}\Big[(S(T) - \frac{1}{T}Y(T))^+ \mid S(t) = x, Y(t) = y\Big]$$

Note $Y(T) = Y(t) + \int_0^T S(u)\,du$, and $S(u) = S(t)e^{\sigma(\widetilde{W}(u)-\widetilde{W}(t))+(r-\sigma^2/2)(u-t)}$ for $u > t$. For notational convenience, let $J(t,u) = e^{\sigma(\widetilde{W}(u)-\widetilde{W}(t))+(r-\sigma^2/2)(u-t)}$, and observe it is independent of $\mathcal{F}(t)$. Thus

$$\begin{aligned} v(t,x,y) &= e^{-r(T-t)}\tilde{E}\Big[(S(t)J(t,T) - \frac{1}{T}[Y(t) + S(t)\int_t^T J(t,u)\,du])^+ \mid S(t) = x, Y(t) = y\Big] \\ &= e^{-r(T-t)}\tilde{E}\Big[\Big(xJ(t,T) - \frac{1}{T}[y + x\int_t^T J(t,u)\,du]\Big)^+\Big] \end{aligned}$$

This formula makes sense for all y, positive and negative.

b) Find a p.d.e. for v. Be sure to specify the domain of (t, x, y)-space where the equation holds. Derive or write down all boundary and terminal conditions. You will have to specify a boundary condition that fixes the value of v, either as $y \to \infty$ or $y \to -\infty$.

Ans: Since $\lim_{y\to\infty} \left(xJ(t,T) - \frac{1}{T}[y + x\int_t^T J(t,u)\,du]\right)^+ = 0$ with probability one, it follows from the formula of the part a) that

$$\lim_{y\to\infty} v(t,x,y) = 0.$$

If $x = 0$, then, from the formula of part a), $v(t,x) = e^{-r(T-t)}\tilde{E}[(-y/T)^+] = e^{-r(T-t)}(-y/T)^+ = T^{-1}e^{-r(T-t)}|y|\mathbf{1}_{\{y<0\}}$.

We shall consider the more general problem in which we allow $Y(t)$ to have any starting value y, not just $Y(0) = 0$ as in the original formulation. Then $v(t, S(t), Y(t))$ is still the price of the option and $e^{-r(T-t)}v(t, S(t), Y(t))$ must be a martingale. We apply Itô's rule to this and set the 'dt' integrand to zero. The result is

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y)$$

on the domain $0 \le t < T$, $x \ge 0$, $-\infty < y < \infty$. The boundary conditions we have derived already and are

$$\lim_{y\to\infty} v(t,x,y) = 0, \qquad \text{for } x \ge 0 \text{ and } 0 \le t < T \text{ and}$$
$$v(t,0,y) = T^{-1}e^{-r(T-t)}|y|\mathbf{1}_{\{y<0\}} \qquad \text{for } 0 \le t < T.$$

The terminal condition is $v(T,x,y) = \left(x - (y/T)\right)^+$.

2. Extra Credit (5 points) Let $C(t)$ be the price of a European call with strike K expiring at time T. Let $A(t)$ be the price of the corresponding American call. No particular model is put on the price process, except that we assume no dividends are paid on the asset. Use a no-arbitrage argument to show that $C(t) > (S_t - K)^+$ for all $t < T$, as long as S_T can fall to either side of K with positive probability. Use this result to show that $A(t) = C(t)$ for all t and that early exercise is never optimal. (It is also helpful to keep this observation in mind: If $A(t) > (S_t - K)^+$ then it is not optimal to exercise the American option at time t since trading the option itself gives higher pay off than exercising the option).

Ans: As long as $P(S(T) > K) > 0$, the price $C(t) > 0$, because the probability of a strictly positive payoff is greater than zero.

If $0 < C(t) \le (S(t) - K)^+$ at some $t < T$, then $S(t) > K$ and $S(t) \ge C(t) + K$. This would create an arbitrage opportunity. Suppose you short one share of the underlying (that is you borrow $S(t)$ in cash from an agent and pay back one share of S at time T - Another way to think about it is you borrow 1 share of S now and pay it back at time T) and buy the European call for $C(t)$, this leaves you with at least $S(t) - C(t) \ge K$ to invest at the risk-free rate. Since you owe a share of the underlying, your return from this position at T is

$$\begin{aligned}(S(t) - C(t))e^{r(T-t)} + (S(T) - K)^+ - S(T) &= (S(t) - C(t))e^{r(T-t)} - \min\{K, S(T)\} \\ &\ge Ke^{r(T-t)} - \min\{K, S(T)\}\end{aligned}$$

If $r > 0$, this is always positive, and so it yields an arbitrage. If $r = 0$, this payoff is non-negative and strictly positive on the event $S(t) < K$, which we assume happens with positive probability, so again we have an arbitrage. It follows that $C(t) > (S(t) - K)^+$, if $t \le T$, in order that there is no arbitrage.

The price of the, $A(t)$ of the American call is always greater than or equal to $C(t)$. Thus $A(t) \ge C(t) > (S(t) - K)^+$ when $t < T$. Since the value of the American call is thus always strictly greater than the value of immediate exercise if $t < T$, it is optimal to exercise the American call only at T. It follows then that the American and European call have the same value: $A(t) = C(t)$, for all $t \le T$.