Homework 5 (Due 3/9/16)

March 5, 2016

1. Let W be a Brownian motion and let $\{\mathcal{F}(t);\, t \geq 0\}$ be filtration for W. We claimed in the lecture note (Theorem 4.2.16) that if $Y(t) = \int_0^t \alpha(s)\, dW(s)$, and if τ is a stopping time with respect to $\{\mathcal{F}(t);\, t \geq 0\}$, then $Y(t \wedge \tau) = \displaystyle\int_0^t \mathbf{1}_{[0,\tau)}(s)\alpha(s)\, dW(s)$.

In this problem we want to show a special case of this. Assume that $\tau(\omega) \leq T$ for all ω where T is positive constant. We want to show

$$W(\tau) = \int_0^T \mathbf{1}_{[0,\tau)}(s)\, dW(s) \tag{1}$$

a) Case (i): The stopping time τ takes values in a discrete set $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$. In this case, identify random variables $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that α_k is $\mathcal{F}(t_k)$-measurable for each k, and

$$\mathbf{1}_{[0,\tau)}(s) = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{[t_k, t_{k+1})}(s).$$

This shows that $\mathbf{1}_{[0,\tau)}(s)$ is a simple process as defined in Shreve, section 4.2.1. (What we call α_k here is what is denoted by $\triangle(t_k)$ in §4.2.1.) Now apply the definition of the stochastic integral for simple processes in equation (4.2.2) to prove (1).

b) Case (ii). The general case. Let τ be any stopping time with $\tau(\omega) \leq T$ for all ω. Let $\tau^{(n)}$ be the approximation to τ constructed in part (a) of exercise 3. Since $\tau^{(n)}$ takes values in a discrete set, equation (1) is true when τ is replace by $\tau^{(n)}$, for each n. Argue that $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)$ for all ω, and conclude that (1) is true for τ.

2. Consider the two asset, risk-neutral model

$$\begin{aligned} dS_1(t) &= rS_1(t)\, dt + \sigma_1(S_1(t), S_2(t))S_1(t)\, d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\, dt + \sigma_2(S_1(t), S_2(t))S_2(t)\, d\widetilde{W}_2(t) \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions and $\sigma_1(x_1, x_2)$ and $\sigma_2(x_1, x_2)$ are strictly positive, bounded, differentiable functions. You may take as known that, given $S_1(0)$ and $S_2(0)$, this system has a unique solution which is a Markov process. Let τ be the first time that $S_1(t)$ hits the level $B > 0$, and let ρ be the first time $S_2(t)$ hits B. Consider an option which knocks out if either $S_1(t)$ hits B or $S_2(t)$ hits B, and otherwise pays $(S_1(T)S_2(T) - K)^+$ at time T. Denote its price by $V(t)$.

a) Show that $V(t) = \mathbf{1}_{\{\tau \wedge \rho > t\}} v(t, S_1(t), S_2(t))$, where $v(t, x_1, x_2) = 0$ if $x_1 \geq B$ or $x_2 \geq B$, and otherwise,

$$v(t, x_1, x_2) = e^{-r(T-t)} \tilde{E}\Big[\quad \mathbf{1}_{\max_{[t,T]} S_1(u) < B} \mathbf{1}_{\max_{[t,T]} S_2(u) < B} \Big(S_1(T)S_2(T) - K\Big)^+ \Big| S_1(t) = x_1, S_2(t) = x_2 \Big].$$

b) Show that $e^{-r(t \wedge \tau \wedge \rho)} v(t, S_1(t \wedge \tau \wedge \rho), S_2(t \wedge \tau \wedge \rho))$ is a martingale and derive a partial differential equation for $v(t, x_1, x_2)$. Specify the domain in (x_1, x_2)-space on which this equation is valid and all boundary and terminal conditions.