Homework 5 (Sol)

March 10, 2016

1. Let W be a Brownian motion and let $\{\mathcal{F}(t);\, t \geq 0\}$ be filtration for W. We claimed in class that if $Y(t) = \int_0^t \alpha(s)\, dW(s)$, and if τ is a stopping time with respect to $\{\mathcal{F}(t);\, t \geq 0\}$, then $Y(t \wedge \tau) = \displaystyle\int_0^t \mathbf{1}_{[0,\tau)}(s)\alpha(s)\, dW(s)$.

In this problem we want to show a special case of this. Assume that $\tau(\omega) \leq T$ for all ω where T is positive constant. We want to show

$$W(\tau) = \int_0^T \mathbf{1}_{[0,\tau)}(s)\, dW(s) \tag{1}$$

a) Case (i): The stopping time τ takes values in a discrete set $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$. In this case, identify random variables $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that α_k is $\mathcal{F}(t_k)$-measurable for each k, and

$$\mathbf{1}_{[0,\tau)}(s) = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{[t_k, t_{k+1})}(s).$$

This shows that $\mathbf{1}_{[0,\tau)}(s)$ is a simple process as defined in Shreve, section 4.2.1. (What we call α_k here is what is denoted by $\triangle(t_k)$

Ans:

Note that

$$\begin{aligned} \{0 \leq s < \tau\} &= \cup_{k=0}^{n-1}\{0 \leq s < \tau, t_k \leq s < t_{k+1}\} \\ &= \cup_{k=0}^{n-1}\{\tau > t_k, t_k \leq s < t_{k+1}\}. \end{aligned}$$

Indeed, if $t_k \leq s < t_{k+1}$ and $s < \tau$ then it must be the case that $\tau > t_k$. Conversely, if $t_k \leq s < t_{k+1}$ and $\tau > t_k$ then since τ takes values only in t_k's, it must be that $\tau \geq t_{k+1}$ and thus $s < \tau$. So $\alpha_k = \mathbf{1}_{\tau > t_k}$. Note that $\alpha_k \in \mathcal{F}(t_k)$ as the requirement for simple function for definition of stochastic integral as well.

Thus

$$\begin{aligned}
\int_0^T \mathbf{1}_{[0,\tau)}(s)dW(s) &= \sum_{k=0}^{n-1} \mathbf{1}_{\tau>t_k}\mathbf{1}_{[t_k,t_{k+1})}(s)dW(s) \\
&= \sum_{k=0}^{n-1} \mathbf{1}_{\tau>t_k} W(t_{k+1}) - W(t_k) \\
&= \sum_{k=0}^{\tau-1} \mathbf{1}_{\tau>t_k} W(t_{k+1}) - W(t_k) \\
&= \sum_{k=0}^{\tau-1} W(t_{k+1}) - W(t_k) \\
&= W(\tau).
\end{aligned}$$

b) Case (ii). The general case. Let τ be any stopping time with $\tau(\omega) \leq T$ for all ω. Let $\tau^{(n)}$ be the approximation to τ constructed in part (a) of exercise 3. Since $\tau^{(n)}$ takes values in a discrete set, equation (1) is true when τ is replace by $\tau^{(n)}$, for each n. Argue that $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)$ for all ω, and conclude that (1) is true for τ.

Ans:

It is clear that

$$|\tau_n - \tau| \leq \frac{1}{n}.$$

Thus $\tau_n \to \tau$ as $n \to \infty$. Because $W(t)$ is continuous, it follows that

$$W(\tau_n) \to W(\tau).$$

Also

$$E(\int_0^T |\mathbf{1}_{[0,\tau_n)}(s) - \mathbf{1}_{[0,\tau)}(s)|^2 ds) \to 0 \text{ as } n \to \infty.$$

Thus

$$\int_0^T \mathbf{1}_{[0,\tau_n)}(s)dW(s) \to \int_0^T \mathbf{1}_{[0,\tau)}(s)dW(s).$$

By identifying the limits, it follows that

$$W(\tau) = \int_0^T \mathbf{1}_{[0,\tau)}(s)dW(s).$$

2. Consider the two asset, risk-neutral model

$$\begin{aligned} dS_1(t) &= rS_1(t)\,dt + \sigma_1(S_1(t),S_2(t))S_1(t)\,d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\,dt + \sigma_2(S_1(t),S_2(t))S_2(t)\,d\widetilde{W}_2(t) \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions and $\sigma_1(x_1,x_2)$ and $\sigma_2(x_1,x_2)$ are strictly positive, bounded, differentiable functions. You may take as known that, given $S_1(0)$ and $S_2(0)$, this system has a unique solution which is a Markov process. Let τ be the first time that $S_1(t)$ hits the level $B>0$, and let ρ be the first time $S_2(t)$ hits B. Consider an option which knocks out if either $S_1(t)$ hits B or $S_2(t)$ hits B, and otherwise pays $(S_1(T)S_2(T)-K)^+$ at time T. Denote its price by $V(t)$.

a) Show that $V(t)=\mathbf{1}_{\{\tau\wedge\rho>t\}}v(t,S_1(t),S_2(t))$, where $v(t,x_1,x_2)=0$ if $x_1\geq B$ or $x_2\geq B$, and otherwise,

$$v(t,x_1,x_2)=e^{-r(T-t)}\tilde{E}\Big[\mathbf{1}_{\max_{[t,T]}S_1(u)<B}\mathbf{1}_{\max_{[t,T]}S_2(u)<B}\Big(S_1(T)S_2(T)-K\Big)^+\Big|S_1(t)=x_1,S_2(t)=x_2\Big].$$

Ans: We have

$$V(T)=\mathbf{1}_{\max_{[0,T]}S_1(u)<B}\mathbf{1}_{\max_{[0,T]}S_2(u)<B}\Big(S_1(T)S_2(T)-K\Big)^+.$$

Thus

$$V(t)=\mathbf{1}_{\max_{[0,t]}S_1(u)<B}\mathbf{1}_{\max_{[0,t]}S_2(u)<B}\tilde{E}\quad\Big(\quad\mathbf{1}_{\max_{[t,T]}S_1(u)<B}\mathbf{1}_{\max_{[t,T]}S_2(u)<B}\Big(S_1(T)S_2(T)-K\Big)^+|\mathcal{F}(t)\Big)$$

or

$$V(t)=\mathbf{1}_{\max_{[0,t]}S_1(u)<B}\mathbf{1}_{\max_{[0,t]}S_2(u)<B}E\quad\Big(\quad\mathbf{1}_{\max_{[t,T]}S_1(u)<B}\mathbf{1}_{\max_{[t,T]}S_2(u)<B}\Big(S_1(T)S_2(T)-K\Big)^+|S_1(t),S_2(t)\Big),$$

by Markov property of S_1,S_2. Note that

$$\mathbf{1}_{\max_{[0,t]}S_1(u)<B}\mathbf{1}_{\max_{[0,t]}S_2(u)<B}=\mathbf{1}_{\{\tau\wedge\rho>t\}}.$$

Indeed, if the LHS $=1$ then neither S_1 nor S_2 has hit B before or at time t. Thus the minimum of the hitting time to b of S_1, S_2 must be $>t$ and so the RHS $=1$. On the other hand, if LHS $=0$, then either S_1 or S_2 has hit B before or at time t. But then either ρ or τ is $<=t$ thus the RHS is also 0.

b) Show that $e^{-r(t\wedge\tau\wedge\rho)}v(t,S_1(t\wedge\tau\wedge\rho),S_2(t\wedge\tau\wedge\rho))$ is a martingale and derive a partial differential equation for $v(t,x_1,x_2)$. Specify the domain in (x_1,x_2)-space on which this equation is valid and all boundary and terminal conditions.

Ans: We need to show

$$v(t,S_1(t\wedge\tau\wedge\rho),S_2(t\wedge\tau\wedge\rho)) = \mathbf{1}_{\{\tau\wedge\rho>t\}}v(t,S_1(t),S_2(t))$$

If $\tau\wedge\rho>t$ then RHS $=v(t,S_1(t),S_2(t))$ and it is clear that the LHS $=v(t,S_1(t),S_2(t))$ as well.

If $\tau\wedge\rho<t$ then RHS $=0$ and either

$$S_1(t\wedge\tau\wedge\rho) \;=\; S_1(\tau\wedge\rho)=B$$

or

$$S_2(t\wedge\tau\wedge\rho) \;=\; S_2(\tau\wedge\rho)=B.$$

Suppose the first case is true. By the immediate crossing property of S_1 we have

$$\begin{aligned} v(t,S_1(t\wedge\tau\wedge\rho),S_2(t\wedge\tau\wedge\rho)) &= e^{-r(T-t)}\tilde{E}\Big[\mathbf{1}_{\max_{[t,T]}S_1(u)<B}\mathbf{1}_{\max_{[t,T]}S_2(u)<B}\Big(S_1(T)S_2(T)-K\Big)^+ \\ &\qquad \Big|S_1(t)=B,S_2(t)=x_2\Big]\Big|_{x_2=S_2(\tau\wedge\rho)} \\ &= 0. \end{aligned}$$

The other case is similar. So we also have the LHS $=0$.

Now denote $\rho\wedge\tau:=\bar{\tau}$ to simplify notation and apply Itos' formula, we have

$$\begin{aligned} e^{-r(t\wedge\bar{\tau})}v(t\wedge\bar{\tau},S_1(t\wedge\bar{\tau}),S_2(t\wedge\bar{\tau})) &= v(t,S_1(0),S_2(0)) \\ &+\int_0^{t\wedge\bar{\tau}} e^{-ru}v_{x_1}(u,S_1(u),S_2(u))\,\sigma_1(S_1(u),S_2(u))S_1(u)\,d\widetilde{W}_1(u) \\ &+\int_0^{t\wedge\bar{\tau}} e^{-ru}v_{x_1}(u,S_1(u),S_2(u))\,\sigma_1(S_1(u),S_2(u))S_1(u)\,d\widetilde{W}_2(u) \\ &+\int_0^{t\wedge\bar{\tau}} e^{-ru}\mathcal{L}v(u,S_1(u),S_2(u))\,du, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}v(t,x_1,x_2) &= -rv(t,x_1,x_2)+v_t(t,x_1,x_2)+rx_1v_{x_1}(t,x_1,x_2)+rx_2v_{x_2}(t,x_1,x_2) \\ &\quad +[\sigma_1^2(x_1,x_2)x_1^2/2]v_{x_1x_1}(t,x_1,x_2)+[\sigma_2^2(x_1,x_2)x_2^2/2]v_{x_2x_2}(t,x_1,x_2). \end{aligned}$$

In order that the previous expression be a martingale, the ordinary integral should be identically 0, and this can be achieved by setting

$$\mathcal{L}v(t, x_1, x_2) = 0 \qquad \text{for } 0 \le t < T,\ 0 \le x_1 < B,\ 0 \le x_2 < B.$$

The domain is important. In the integral $\int_0^{t\wedge\bar{\tau}} \mathcal{L}v(u, S_1(u), S_2(u))\, du$, the process $(S_1(u), S_2(u))$ remains in $\{0 < x_1 < B, 0 < x_2 < B\}$ until $t \wedge \bar{\tau}$, so we want and only need to assume the existence and continuity of first and second derivatives of v in this region.

The boundary and terminal conditions are $v(t, 0, x_2) = v(t, x_1, 0) = 0$ for $0 \le t < T$; $v(t, x_1, B) = v(t, B, x_2) = 0$ for $0 \le x_1, x_2 \le B$ and $t \le T$, and $v(T, x_1, x_2) = (x_1 x_2 - K)^+$ for $0 \le x_1, x_2 < B$.