Homework 4 (Due 3/2/2016)

Math 622

February 25, 2016

1. Let τ and ρ be stopping times with respect to a filtration $\{\mathcal{F}(t);\, t \geq 0\}$.

a) Show that $\tau \wedge \rho \,(= \min\{\tau, \rho\})$ is a stopping time.

b) Show that $\tau \vee \rho \,(= \max\{\tau, \rho\})$ is a stopping time.

2. Let $X = \{X(t);\, t \geq 0\}$ be a stochastic process whose sample paths are all continuous, and assume $X(0)(\omega) = 0$ for all ω. Let $\{\mathcal{F}^X(t)\}_{t \geq 0}$ be the filtration generated by X.

In each case below, determine whether the random time must necessarily be an $\{\mathcal{F}^X(t);\, t \geq 0\}$-stopping time. Justify your answer briefly in each case; you may use the informal rule of thumb and/or general results about stopping times, as in the Lecture Notes for lecture 4.

(a) $T_1 = \inf\{t;\, X^2(t) \geq 1\}$;

(b) $T_2 = \inf\{t;\, \int_0^t X^2(s)\, ds > 1\}$.

(c) $T_3 = \sup\{t;\, t \leq 1 \text{ and } X(t) = 0\}$;

(d) Suppose that $|X(t)|(\omega) > 0$ for all $\omega \in \Omega$ and all $t > 0$, and reconsider $T_2 = \inf\{t;\, \int_0^t X^2(s)\, ds > 1\}$.

(e) $T_5 = \inf\{t; X(t) \geq X(t+1)\}$.

3. Let τ be a stopping time with respect to a filtration, $\{\mathcal{F}(t);\, t \geq 0\}$.

a) Let n be any positive integer. Define a discrete approximation $\tau^{(n)}$ to τ by setting $\tau^{(n)}(\omega) = \frac{k}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. This approximates τ from above. Show that $\tau^{(n)}$ is an $\{\mathcal{F}(t);\, t \geq 0\}$-stopping time.

b) Let n be any positive integer and define a discrete approximation to τ from below by $\tau_n(\omega) = \frac{k-1}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. Is τ_n in general an $\{\mathcal{F}(t);\, t \geq 0\}$-stopping time? Explain.

4. (Optional Stopping) Let $\{X_n\}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}$; thus (i) X_n is $\mathcal{F}_n$-measurable for each n, (ii) $E[|X_n|] < \infty$ for each n, and (iii) $E[X_{n+1} \big| \mathcal{F}_n] = X_n$ for each n. Let τ be a stopping time with respect to $\{\mathcal{F}_n\}$. Show that the stopped process $X_{n\wedge\tau}$ is also a martingale with respect to $\{\mathcal{F}_n\}$.

Hint: Write $X_{n\wedge\tau} = \sum_{k=0}^{n} X_k \mathbf{1}_{\{\tau=k\}} + X_n \mathbf{1}_{\{\tau>n\}}$. Observe that $\{\tau > n\}$ is $\mathcal{F}_n$-measurable (why?).