Homework 4 (Sol)

Math 622

March 3, 2016

1. Let τ and ρ be stopping times with respect to a filtration $\{\mathcal{F}(t);\, t \geq 0\}$.

a) Show that $\tau \wedge \rho\, (= \min\{\tau, \rho\})$ is a stopping time.

$$\{\tau \wedge \rho > t\} = \{\tau > t\} \cap \{\rho > t\} \in \mathcal{F}(t),$$

since both τ and ρ are stopping times. It follows that

$$\{\tau \wedge \rho \leq t\} = \{\tau \wedge \rho > t\}^c \in \mathcal{F}(t),$$

and $\tau \wedge \rho$ is a stopping time.

b) Show that $\tau \vee \rho\, (= \max\{\tau, \rho\})$ is a stopping time.
Similar to part a:

$$\{\tau \vee \rho \leq t\} = \{\tau \leq t\} \cap \{\rho \leq t\} \in \mathcal{F}(t).$$

Thus $\tau \vee \rho$ is a stopping time.

2. Let $X = \{X(t);\, t \geq 0\}$ be a stochastic process whose sample paths are all continuous, and assume $X(0)(\omega) = 0$ for all ω. Let $\{\mathcal{F}^X(t)\}_{t \geq 0}$ be the filtration generated by X.

(a) $T_1 = \inf\{t;\, X^2(t) \geq 1\}$;

Ans: Note that $T_1 = \inf\{t;\, X^2(t) = 1\}$ since $X(0) = 0$ therefore it is a stopping time.

(b) $T_2 = \inf\{t;\, \int_0^t X^2(s)\, ds > 1\}$.

Ans: It is very tempting to quote the example in the lecture note to say T_2 may not be a stopping time. But the situation is slightly different here (see also part d). Denote $Y_t = \int_0^t X^2(s)ds$ then Y_t is differentiable in t, moreover, $Y'(t) = X^2(t) \geq 0$. So it cannot be the case that $Y(t)$ just touches 1 at time t and bounces back. BUT it is possible that $Y(t)$ touches 1 at time t and stays at 1 for a short interval, say $[t, t+\epsilon)$, instead of crossing 1 at t. In the first senario, $T_2 = t + \epsilon$ and second, $T_2 = t$. However, up to time t, the sample path of $Y(t)$ is exactly the same, so we cannot determine the event $\{T_2 \leq t\}$ based on observation of $Y_s, 0 \leq s \leq t$. So T_2 may not be a stopping time.

(c) $T_3 = \sup\{t;\ t \leq 1 \ \text{ and } \ X(t) = 0\}$;

May not be a stopping time, same as example given in the note.

(d) Suppose that $|X(t)|(\omega) > 0$ for all $\omega \in \Omega$ and all $t > 0$, and reconsider $T_2 = \inf\{t;\ \int_0^t X^2(s)\, ds > 1\}$.

In this case T_2 IS a stopping time, since the senario of staying at 1 on $[t, t+\epsilon)$ is no longer possible (which would imply $Y'(s) = X^2(s) = 0, s \in [t, t+\epsilon)$ contradicting the hypothesis that $|X(s)| \neq 0$.) So $T_2 = \inf\{t \geq 0 : Y(t) = 1\}$ which is a stopping time.

(e) $T_5 = \inf\{t; X(t) \geq X(t+1)\}$.

T_5 may not be a stopping time since if we denote $Y(t) = X(t) - X(t+1)$ then we can only say $Y(t)$ is $\mathcal{F}(t+1)$ measurable. Thus $\{T_5 \leq t\} = \{Y(s) \geq 0, 0 \leq s \leq t\}$ which may not be in $\mathcal{F}(t)$.

3. Let τ be a stopping time with respect to a filtration, $\{\mathcal{F}(t);\ t \geq 0\}$.

a) Let n be any positive integer. Define a discrete approximation $\tau^{(n)}$ to τ by setting $\tau^{(n)}(\omega) = \frac{k}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. This approximates τ from above. Show that $\tau^{(n)}$ is an $\{\mathcal{F}(t);\ t \geq 0\}$-stopping time.

Ans:

$$\begin{aligned}\{\tau_n \leq t\} &= \cup_{k=1}^{\infty}\{\tau_n \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\} \\ &= \cup_{k=1}^{\infty}\{\frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\}.\end{aligned}$$

Note that for each k, the event $\{\frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\} \in \mathcal{F}(t)$. The reason is the event

$$\{\frac{k-1}{n} < \tau \leq \frac{k}{n}\} = \{\frac{k-1}{n} < \tau\} \cap \{\tau \leq \frac{k}{n}\} \in \mathcal{F}(\frac{k}{n}).$$

And the event $\{\frac{k}{n} \leq t\}$ is either Ø or Ω. If it is empty then $\{\frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\} = \emptyset$ and is in $\mathcal{F}(t)$. If it is Ω then it must be the case that $\frac{k}{n} \leq t$ and so $\mathcal{F}(\frac{k}{n}) \subseteq \mathcal{F}(t)$ and we reach the same conclusion.

Hence $\{\tau_n \leq t\}$ being countable union of event in $\mathcal{F}(t)$ is in $\mathcal{F}(t)$ and τ_n is a stopping time.

b) Let n be any positive integer and define a discrete approximation to τ from below by $\tau_n(\omega) = \frac{k-1}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. Is τ_n in general an $\{\mathcal{F}(t);\, t \geq 0\}$-stopping time? Explain.

Ans: τ_n may not be a stopping time. Arguing similar to the above, we get

$$\begin{aligned}
\{\tau_n \leq t\} &= \cup_{k=1}^{\infty}\{\tau_n \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\} \\
&= \cup_{k=1}^{\infty}\{\frac{k-1}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\}.
\end{aligned}$$

Note that now we deal with the event $\{\frac{k-1}{n} \leq t\}$ while it is still the case that the event $\{\frac{k-1}{n} < \tau \leq \frac{k}{n}\} \in \mathcal{F}(\frac{k}{n})$. But if $\frac{k-1}{n} \leq t$ it does not necessarily follow that $\frac{k}{n} \leq t$ and thus the event $\{\frac{k-1}{n} < \tau \leq \frac{k}{n}\}$ may not be in $\mathcal{F}(t)$.

4. (Optional Stopping) Let $\{X_n\}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}$; thus (i) X_n is $\mathcal{F}_n$-measurable for each n, (ii) $E[|X_n|] < \infty$ for each n, and (iii) $E[X_{n+1}\big|\mathcal{F}_n] = X_n$ for each n. Let τ be a stopping time with respect to $\{\mathcal{F}_n\}$. Show that the stopped process $X_{n\wedge\tau}$ is also a martingale with respect to $\{\mathcal{F}_n\}$.

Ans: We only need to show for all n

$$E(X_{n\wedge\tau}|\mathcal{F}(n-1)) = X_{n-1\wedge\tau}.$$

Note that

$$X_{n\wedge\tau} = \sum_{k=0}^{n-1} X_k \mathbf{1}_{\{\tau=k\}} + X_n \mathbf{1}_{\{\tau\geq n\}}.$$

and all events $\{\tau = k\}, k = 0, ..., n-1$ and $\{\tau \geq n\}$ is in $\mathcal{F}(n-1)$. Only the last one needs explanation. Note that $\{\tau \geq n\} = \{\tau < n\}^c = \left(\cup_{k=1}^{n-1}\{\tau = k\}\right)^c$ so it is in $\mathcal{F}(n-1)$. Thus

$$\begin{aligned}
E(X_{n\wedge\tau}|\mathcal{F}(n-1)) &= \sum_{k=0}^{n-1} X_k \mathbf{1}_{\{\tau=k\}} + \mathbf{1}_{\{\tau\geq n\}} E(X_n|\mathcal{F}(n-1)) \\
&= \sum_{k=0}^{n-1} X_k \mathbf{1}_{\{\tau=k\}} + \mathbf{1}_{\{\tau\geq n\}} X_{n-1} \\
&= \sum_{k=0}^{n-2} X_k \mathbf{1}_{\{\tau=k\}} + \mathbf{1}_{\{\tau\geq n-1\}} X_{n-1} = X_{n-1\wedge\tau}.
\end{aligned}$$