

Homework 3 (Due 2/24/2016)

Math 622

February 22, 2016

1. On a probability space (Ω, \mathbb{P}) let N_1, N_2 be independent Poisson processes with rate λ_1, λ_2 and $\mathcal{F}(t)$ a filtration for N_1, N_2 . Assume $b_1 > 0 > b_2 > -1$ and let

$$Q(t) := b_1 N_1(t) + b_2 N_2(t).$$

- (i) Find m so that $M(t) = Q(t) - mt$ is a $\mathcal{F}(t)$ -martingale.
- (ii) Consider the price model

$$dS(t) = \alpha S(t)dt + S(t-)dM(t), S(0) = 1.$$

Write down a solution in the form $S(t) = Ke^{a_0 t + a_1 N_1(t) + a_2 N_2(t)}$; identify the constants a_0, a_1, a_2 and K .

(iii) Suppose now that $a \neq r$ for the model in part (ii). Show how we can define a risk-neutral measure \mathbb{Q} for the model in (ii).

(iv) Show that there are in fact many different risk-neutral measures for the setting in (iii).

(v) Suppose that now we consider a market with 2 assets:

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t)dt + S_1(t-)dM(t), S_1(0) = 1 \\ dS_2(t) &= \alpha_2 S_2(t)dt + \sigma_2 S_2(t-)dN_1(t), S_2(0) = 1, \end{aligned}$$

where $\sigma_2 > 0$. A risk neutral probability \mathbb{Q} for this market must be such that $e^{-rt}S_1(t)$ and $e^{-rt}S_2(t)$ are $\mathcal{F}(t)$ martingales under \mathbb{Q} . Show how we can define a risk neutral measure \mathbb{Q} for this market. What conditions must $\alpha_1, \alpha_2, \sigma_2, \lambda_1, \lambda_2, r$ satisfy for this measure change to be valid? When is \mathbb{Q} unique? (It is helpful to look at the discussion in Shreve's page 516).

2. *Merton's jump diffusion process.* Let Z_1, Z_2, \dots be independent standard normal random variables that are independent of W and N , where W and N are independent,

W is a Brownian motion, and N is a Poisson process with rate λ . Let $Q(t) = \sum_1^{N(t)} [e^{Z_i} - 1]$. Consider the price model

$$dS(t) = \alpha S(t) dt + S(t)\sigma dW(t) + S(t-)dQ(t),$$

This price process is called Merton's jump diffusion.

a) Explicitly identify a constant θ and a compound Poisson process \bar{Q} such that

$$S(t) = S(0) \exp\{\sigma W(t) + \theta t + \bar{Q}(t)\}.$$

b) Let r be the risk-free interest rate. What must α be so that this model is risk-neutral?

3. Let $Q(t)$ be the compound Poisson process

$$Q(t) = \sum_{k=1}^{N(t)} Y_k,$$

where Y_1, Y_2, \dots are i.i.d. with $\mathbf{P}\left(Y_i = \frac{3}{4}\right) = \frac{3}{5}$ and $\mathbf{P}\left(Y_i = -\frac{3}{4}\right) = \frac{2}{5}$, and where N is a Poisson process with rate 2. Let $N_1(t)$ count the number of jumps of Q by $3/4$ and let $N_2(t)$ count the number of jumps of Q by $-3/4$.

Consider,

$$dS(t) = -(3/10)S(t) dt + S(t-) dQ(t), \quad S(0) = 1. \quad (1)$$

If S solves equation (1), is it a martingale or not? Explain.

4. Read Corollary 11.5.3. The following problem is to prove a similar result for two Poisson processes rather than a Poisson process and a Brownian motion. Let N_1 and N_2 be Poisson processes relative to the same filtration. Let their rates be λ_1 and λ_2 . Assume they have no simultaneous jumps. Show they are independent, as follows. Show that the calculation of problem 2 of Assignment 2 is valid and deduce from it that the process defined there is a martingale for each u_1 and u_2 .

5. Create a price model for a single asset price with the following properties:

- (i) Normally the price follows a Black-Scholes type of evolution with a constant volatility.

- (ii) Occasionally the price jumps up by an amount that is, on average, one quarter of the pre-jump price. These jumps arrive according to a Poisson process.
- (iii) Occasionally the price jumps down by an amount that is, on average, one third of the pre-jump price. These price jolts arrive according to a Poisson process independently of the positive jumps.
- (iv) The positive jumps arrive at a faster rate than the negative jumps.

There is no one right answer to this problem. Your model will contain different parameters. Try to leave as many as possible as free constants—you would want to be able to choose these parameters to fit empirical data if you were to implement the model. However, the conditions (i)—(iv) might imply relation(s) among the parameters and you should specify these. *You may want to utilize this fact in the construction of the model: the sum of 2 compound Poisson processes is still a compound Poisson process in the following sense: if $N_1(t), N_2(t)$ are independent Poisson processes with rates λ_1, λ_2 , X_i i.i.d and Y_i i.i.d random variables such that $\{X_i\}$ is independent of $\{Y_i\}$ then*

$$Q(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{j=1}^{N_2(t)} Y_j$$

is also a compound Poisson process. Can you see why? Can you write $Q(t)$ in the regular form of a Compound Poisson process?